Discrete Optimization
ISyE 6662 - Spring 2023
Homework 1

Instructor: Alejandro Toriello
TA: Filipe Cabral

January 23, 2023.

1. Let $A \in Z^{n\times n}$ be a square integer matrix, and suppose the largest absolute value of any entry in A is a. Prove that $|\det(A)| \leq (an)^n$.

   Answer: We know that the determinant function is given by $\det(A) = \sum_{\sigma\in S_n} \text{sgn}(\sigma)a_{1,\sigma(1)}\cdots a_{n,\sigma(n)}$, where S_n is the set of all permutations $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ and $\text{sgn}(\sigma) \in \{-1,1\}$ is the sign of the permutation σ. Thus, the following inequalities hold:

   $$|\det(A)| \leq \sum_{\sigma\in S_n} |a_{1,\sigma(1)}\cdots a_{n,\sigma(n)}| \leq \sum_{\sigma\in S_n} a^n \leq n!a^n \leq n^n a^n,$$

   where the first follows from the triangular inequality, the second is because a is the largest absolute entry of A, and the third is because the number of all possible permutations of n objects is $n!$.

2. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a non-empty polyhedron. Prove the following:

   a) P contains a line if and only if $Ax = 0$ has a non-zero solution.

   Answer: Suppose that P contains a line, i.e., there exists $d \in \mathbb{R}^n\backslash\{0\}$ and $x \in P$ such that $x + td$ belongs to P for all $t \in \mathbb{R}$. Suppose that $(Ad)_i \neq 0$ for some $i = 1,\dots,m$. Let $t_\alpha = \alpha\cdot(Ad)_i$ and note that

   $$b_i \geq (Ax + t_\alpha Ad)_i = (Ax)_i + \alpha\cdot(Ad)_i^2, \quad \forall\alpha > 0.$$

   However, this is a contradiction since $(Ax)_i + \alpha\cdot(Ad)_i^2$ tends to $+\infty$ as α goes to $+\infty$. Thus, d is a non-zero solution to $Ax = 0$.

   Conversely, suppose that $Ax = 0$ has a non-zero solution $d \in \mathbb{R}^n\backslash\{0\}$. Then, $x + td$ belongs to P for every $x \in P$ and $t \in \mathbb{R}$ because

   $$A(x + td) = Ax + t\cdot\underbrace{Ad}_{=0} = Ax \leq b.$$

   b) P is unbounded if and only if $Ax \leq 0$ has a non-zero solution.

   Answer: Suppose that P is unbounded, that is, there exists $d \in \mathbb{R}^n\backslash\{0\}$ and $x \in P$ such that $x + td$ belongs to P for all $t \in \mathbb{R}_+$. Here the scalar t is *non-negative* instead of any real number. The remaining argument is similar.

   Suppose that $(Ad)_i > 0$ for some $i = 1,\dots,m$. Let $t_\alpha = \alpha\cdot(Ad)_i$ and note that

   $$b_i \geq (Ax + t_\alpha Ad)_i = (Ax)_i + \alpha\cdot(Ad)_i^2, \quad \forall\alpha > 0.$$

However, this is a contradiction since $(Ax)_i + \alpha \cdot (Ad)_i^2$ tends to $+\infty$ as α goes to $+\infty$. Thus, d is a non-zero solution to $Ax \leq 0$.

Conversely, suppose that $Ax \leq 0$ has a non-zero solution $d \in \mathbb{R}^n \backslash \{0\}$. Then, $x + td$ belongs to P for every $x \in P$ and $t \in \mathbb{R}_+$.

3. Recall that the *convex hull* of a set of points S is the set of points that can be obtained as a convex combination of (finitely points in S, or equivalently, it is the smallest convex set containing S. Let $S = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq \rho^2\}$ for some $\rho \geq 0$. Is $\text{conv}(S \cap \mathbb{Z}^n)$ a polyhedron? What about $\text{conv}(S \cap (\mathbb{R} \times \mathbb{Z}^{n-1}))$, $\text{conv}(S \cap (\mathbb{R}^2 \times \mathbb{Z}^{n-2}))$, and so forth? Justify your answer.

   Answer: We know that a convex combination of union of polytopes is also a polytope (it is just the convex combination of the union of extreme points). Then, we have the following cases:

   (i) $S \cap \mathbb{Z}^n$ is a finite set, so $\text{conv}(S \cap \mathbb{Z}^n)$ is a polytope.

   (ii) $S \cap (\mathbb{R} \times Z^{n-1})$ is a finite union of line segments, so $\text{conv}(S \cap (\mathbb{R} \times Z^{n-1}))$ is also a polytope.

   (iii) $S \cap (\mathbb{R}^k \times Z^{n-k})$ is a finite union of k-dimensional balls for $2 \leq k \leq n$, which implies that $\text{conv}(S \cap (\mathbb{R}^k \times Z^{n-k}))$ is *not* a polytope. For instance, consider $\rho = 1$, $n = 3$ and $k = 2$. Then,

   $$S \cap (\mathbb{R}^2 \times \mathbb{Z}) = \{(0, 0, -1)\} \cup (C_2 \times \{0\}) \cup \{(0, 0, 0)\},$$

   where C_2 is the unit circle given by $C_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$. Thus, the convex set $\text{conv}(S \cap (\mathbb{R}^2 \times \mathbb{Z}))$ is given by the union of two cones connected by the circular base, i.e.,

   $$\text{conv}(S \cap (\mathbb{R}^2 \times \mathbb{Z})) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + |x_3| \leq 1\}.$$

4. Let $P \subseteq \mathbb{R}^n$ be a polytope given as the convex hull of a set of points, $P = \text{conv}\{x_1, \ldots, x_m\}$. Let $c_1, \ldots, c_k \in \mathbb{R}^n$ be a set of objective vectors.

   a) Write a linear program that solves $\min_{x \in P} \max_{l=1,\ldots,k} c^l x$.

   Answer: For a fixed $x \in P$, the term $\max_{l=1,\ldots,k} c^l x$ can be described as the maximum of cx over all the objective costs c subject to $c \in \text{conv}\{c^1, \ldots, c^k\}$. We conclude by taking the dual of this problem and grouping the minimization problems. Indeed,

   $$\begin{aligned}
   \min_{x \in P} \max_{l=1,\ldots,k} c^l x = \ & \min_{x \in P} \max_{\lambda, c} \quad cx \\
   & \qquad \text{s.t.} \quad \textstyle\sum_{l=1}^k \lambda_l c^l = c, \\
   & \qquad \qquad \textstyle\sum_{l=1}^k \lambda_l = 1, \\
   & \qquad \qquad \lambda \geq 0, \ c \in \mathbb{R}^n, \\
   \overset{\text{(dual)}}{=} \ & \min_{x \in P} \min_{\gamma, \pi} \quad \gamma \\
   & \qquad \text{s.t.} \quad c^l \pi + \gamma \geq 0, \qquad l = 1, \ldots, k, \\
   & \qquad \qquad -\pi = x, \\
   & \qquad \qquad \pi \in \mathbb{R}^n, \ \gamma \in \mathbb{R}, \\
   = \ & \min_{\alpha, x, \gamma} \quad \gamma \\
   & \text{s.t.} \quad -c^l x + \gamma \geq 0, \qquad l = 1, \ldots, k, \\
   & \qquad x = \textstyle\sum_{i=1}^m \alpha_i x_i, \\
   & \qquad \textstyle\sum_{i=1}^m \alpha_i = 1, \\
   & \qquad x \in \mathbb{R}^n, \ \gamma \in \mathbb{R}, \ \alpha \geq 0.
   \end{aligned}$$

   b) Explain how you can efficiently solve $\max_{x \in P} \max_{l=1,\ldots,k} c^l x$.

Answer: Note that the following identities hold:

$$\max_{x\in P}\max_{l=1,\ldots,k} c^l x = \max_{l=1,\ldots,k}\max_{x\in P} c^l x = \max_{\substack{i=1,\ldots,m,\\ l=1,\ldots,k}} c^l x_i.$$

Thus, we can find the largest inner product $c^l x_i$ among $l = 1, \ldots, k$ and $i = 1, \ldots, m$.

Now suppose P is given by a set of linear inequalities, $P = \{x \in \mathbb{R}^n : Ax \le b\}$.

c) Repeat question (a).

Answer: A similar idea applies to this question.

$$\begin{aligned}\min_{x\in P}\max_{l=1,\ldots,k} c^l x = \;& \min_{x\in P}\min_{\gamma} \quad \gamma \\ & \quad\text{s.t.} \quad \gamma \ge c^l x, \quad l = 1, \ldots k, \\ & \qquad\qquad \gamma \in \mathbb{R}, \\ = \;& \min_{x\in P}\min_{\gamma} \quad \gamma \\ & \quad\text{s.t.} \quad \gamma \ge c^l x, \qquad l = 1, \ldots k, \\ & \qquad\qquad Ax \le b, \\ & \qquad\qquad x \in \mathbb{R}^n,\ \gamma \in \mathbb{R}.\end{aligned}$$

d) Write a mixed-integer linear program that solves $\max_{x\in P}\max_{l=1,\ldots,k} c^l x$.

Answer: Roughly speaking, we create k copies of the polytope P and use a binary variable $z_l \in \{0, 1\}$ to indicate the copy we refer. Indeed,

$$\begin{aligned}\max_{x\in P}\max_{l=1,\ldots,k} c^l x = \;& \max_{x,x_l,z_l} \quad \textstyle\sum_{l=1}^{k} c^l x_l \\ & \quad\text{s.t.} \quad Ax_l \le z_l b, \qquad l = 1, \ldots, k, \\ & \qquad\qquad \textstyle\sum_{l=1}^{k} z_l = 1, \\ & \qquad\qquad x, x_l \in \mathbb{R}^n,\ z \in \{0, 1\}^k.\end{aligned}$$

Because P is a polytope (bounded polyhedron) the only solution to $Ax \le 0$ is the zero vector. Thus, x_l belongs to P if z_l is 1 and x_l is the zero vector if z_l is 0.

5. Consider an undirected graph (N, E) with associated edge weights $w \in \mathbb{R}^E$, each of which may be positive, negative or zero. For each of the following problems, give an integer programming formulation and prove its correctness.

a) For a given subset of nodes $S \subseteq N$, find the maximum-weight subtree of the graph that contains S and may or may not contain other nodes.

Answer: Let $x_e \in \{0, 1\}$ be the variable that indicates with 1 if the edge $e \in E$ belongs to our subgraph and 0 otherwise. Let y_v be the indicator variable of a node $v \in N$ of our subgraph. First, we require that every node in S belongs to the subgraph, i.e., $y_v = 1$ for every $v \in S$. Also, an edge belongs to the subgraph if both endpoint nodes belong to it, that is, $x_{uv} \le y_u$ and $x_{uv} \le y_v$, for all $u, v \in V$ such that $uv \in E$.

To enforce an acyclic subgraph we consider the cycle elimination constraint $\sum_{e\in E(U)} \le |U| - 1$, for all $U \subset N$ such that $U \ne \emptyset$ and $U \ne N$. In particular, the resulting subgraph is a forest (union of trees). Recall that a forest with n nodes and m connected components have $n - m$ edges. So, we include the constraint $\sum_{e\in E} x_e = \sum_{v\in N} y_v - 1$ to enforce a subtree, that is, only one connected component. Below we have the complete formulation:

$$
\begin{array}{lll}
\max_{x,y} & \sum_{e\in E} w_e x_e & \\
\text{s.t.} & \sum_{e\in E(U)} \le |U|-1, & U\subset N: U\ne\emptyset, N, \\
& \sum_{e\in E} x_e = \sum_{v\in N} y_v - 1, & \\
& x_{uv}\le y_u,\quad x_{uv}\le y_v, & u,v\in V;\ uv\in E \\
& y_v = 1, & v\in S, \\
& x_e\in\{0,1\},\ e\in E. &
\end{array}
$$

b) For a given subset $S \subseteq N$, find the maximum-weight subgraph in which nodes in S have odd degree and nodes in $N \setminus S$ have even degree (including possibly zero).

Answer: We create an auxiliary variable $z_v \in \mathbb{Z}_+$ for each vertex $v \in N$ to represent even and odd degrees, see the formulation below:

$$
\begin{array}{lll}
\max_{x} & \sum_{e\in E} w_e x_e & \\
\text{s.t.} & \sum_{e\in\delta(v)} x_e = 2z_v + 1, & v\in S, \\
& \sum_{e\in\delta(v)} x_e = 2z_v, & v\in N\backslash S, \\
& x_e\in\{0,1\},\ z_v\in\mathbb{Z}_+, & v\in N,\ e\in E.
\end{array}
$$

c) For a positive integer vector $b \in \mathbb{Z}_+^N$, find the maximum-weight graph in which node $i \in N$ has degree b_i.

Answer: This formulation is similar to the one in (b) but without the auxiliary variable:

$$
\begin{array}{lll}
\max_{x} & \sum_{e\in E} w_e x_e & \\
\text{s.t.} & \sum_{e\in\delta(v)} x_e = b_v, & v\in N, \\
& x_e\in\{0,1\}, & e\in E.
\end{array}
$$

d) Find the subset $S \subseteq N$ that maximizes the total weight of edges in $\delta(S)$. Recall that $\delta(S) \subseteq E$ is the set of edges with exactly one endpoint in S.

Answer: Let $x_e \in \{0,1\}$ be the indicator variable of edge e in a cutset $\delta(S)$ and let y_v be the indicator variable of a node v in S. By definition of cutset, the endpoint nodes u and v from an edge $uv \in \delta(S)$ are such that $u \in S$ and $v \in N\backslash S$. Thus, our constraints must enforce that:

i. If $(y_u, y_v) = (1,1)$ or $(0,0)$ then $x_{uv} = 0$.

ii. If $(y_u, y_v) = (1,0)$ or $(0,1)$ then $x_{uv} = 1$.

These properties can be enforced by the constraints $x_{uv} \le y_u + y_v$ and $x_{uv} \le 2 - y_u - y_v$ for all $uv \in E$. Below we present the formulation for the maximum total weight of edges in $\delta(S)$ over all $S \subset N$:

$$
\begin{array}{lll}
\max_{x} & \sum_{e\in E} w_e x_e & \\
\text{s.t.} & x_{uv}\le y_u + y_v, & uv\in E, \\
& x_{uv}\le 2 - y_u - y_v, & uv\in E, \\
& x_e\in\{0,1\},\ y_v\in\{0,1\}, & v\in N,\ e\in E.
\end{array}
$$

6. You are organizing a single-track workshop with n speakers. Your lecture room is available from time 0 to T. Each speaker i has specified the length of their lecture, l_i. Furthermore,because these are prima donna academics, they have also given you an earliest and latest time they want to start, $a_i \le b_i$. (You may assume all numbers are integers.)

a) Write a MIP formulation to determine if you can feasibly arrange the lectures in the room without any overlap. Your formulation should include continuous variables $t_i \in [0, T]$ representing lecture i's start time, and may include other variables.

Answer: Denote by $[p : q]$ the set of consecutive integer from p to q. Let l_0 be equal to 0 and let $\bar{b}_i$ be equal to $\min(b_i, T - l_i)$. Consider the auxiliary variable $z_{ij} \in \{0, 1\}$ that indicates if lecture j starts just after lecture i. Thus, our formulation is given below:

$$
\begin{array}{llll}
\min_{t,z} & 0 & & \\
\text{s.t.} & a_i \leq t_i \leq \bar{b}_i, & i \in [1 : n], & \text{(Start time bounds)} \\
& t_j \geq t_i + l_i - M(1 - z_{ij}), & i \in [0 : n], \quad j \in [1 : n], & \text{(Time ordering)} \\
& \sum_{i=0}^{n} z_{ij} = 1, & j \in [1 : n], & \text{(Exact one lecture before)} \\
& \sum_{j=1}^{n} z_{ij} \leq 1, & i \in [1 : n], & \text{(At most one lecture after)} \\
& t_0 = 0, \quad z_{ii} = 0, & i \in [1 : n], & \text{(Variable elimination)} \\
& t_i \geq 0, \quad z_{ij} \in \{0, 1\}, & i \in [0 : n] \quad j \in [1 : n]. &
\end{array}
$$

b) Because all parameters are integers, we may assume all lectures start at integer times. Write a second formulation using binary indicator variables $x_{it} \in \{0, 1\}$ for $i = 1, \ldots, n$ and $t = 0, \ldots, T$, where $x_{it} = 1$ means lecture i starts at time t.

Answer: Let $\bar{b}_i$ be equal to $\min(b_i, T - l_i)$. The formulation in this case is given as

$$
\begin{array}{llll}
\min_{t,z} & 0 & & \\
\text{s.t.} & \sum_{t=a_i}^{\bar{b}_i} x_{it} = 1, & i \in [1 : n], & \text{(Start time bounds)} \\
& x_{it} = 0, & i \in [1 : n], \ t \in [0 : T] \backslash [a_i : \bar{b}_i], & \text{(Out of bound starts)} \\
& \sum_{\tau=t}^{\min(T, t+l_i-1)} \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{j\tau} \leq n \cdot (1 - x_{it}), & i \in [1 : n], \ t \in [0 : T], & \text{(Conflict elimination)} \\
& x_{it} \in \{0, 1\}, & i \in [1 : n], \ t \in [0 : T]. &
\end{array}
$$

Given a lecture $i \in [1 : n]$, the *conflict elimination* constraint prevents any other lecture $j \neq i$ from start in the time window from t to $t + l_i - 1$ if the lecture i have started at time t, i.e., $x_{it} = 1$.