ISYE 7405

Homework 2

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1. (1 point) Let X be a p-dimensional random vector with covariance matrix Σ, which is of full rank. We know that $\Sigma_{ij} = 0$ implies that X_i and X_j are uncorrelated. Now prove that if $(\Sigma^{-1})_{ij} = 0$, then X_i and X_j have zero partial correlation, i.e., zero correlation in the reminders of X_i and X_j after regressing on the other $p-2$ coordinates.

   Solution: Without loss of generality, we assume $i = 1$ and $j = 2$. We partition the covariance matrix as

$$\Sigma = \begin{pmatrix} \Sigma_{[12]} & \Sigma_{[12][-12]} \\ \Sigma_{[-12][12]} & \Sigma_{[-12][-12]} \end{pmatrix},$$

   where the notation $[-12]$ are the indices other than 1 and 2. The residual of regressing X_1 on $\boldsymbol{X}_{[-12]}$ is

$$e_1 = X_1 - \Sigma_{1[-12]}\Sigma_{[-12]}^{-1}(\boldsymbol{X}_{[-12]} - \boldsymbol{\mu}_{[-12]}),$$

   and the residual of regressing X_2 on $\boldsymbol{X}_{[-12]}$ is

$$e_2 = X_2 - \Sigma_{2[-12]}\Sigma_{[-12]}^{-1}(\boldsymbol{X}_{[-12]} - \boldsymbol{\mu}_{[-12]}).$$

   The covariance between e_1 and e_2 is

$$\begin{aligned} \text{Cov}\,(e_1, e_2) &= \text{Cov}\left(X_1 - \Sigma_{1[-12]}\Sigma_{[-12]}^{-1}(\boldsymbol{X}_{[-12]} - \boldsymbol{\mu}_{[-12]}), X_2 - \Sigma_{2[-12]}\Sigma_{[-12]}^{-1}(\boldsymbol{X}_{[-12]} - \boldsymbol{\mu}_{[-12]})\right) \\ &= \Sigma_{12} - \Sigma_{1[-12]}\Sigma_{[-12]}^{-1}\Sigma_{[-12]2}. \end{aligned}$$

   According to the formula of the inverse of block matrix, we have

$$\Sigma^{-1} = \begin{pmatrix} (\Sigma^{-1})_{[12][12]} & * \\ * & * \end{pmatrix},$$

   where

$$(\Sigma^{-1})_{[12][12]}^{-1} = \Sigma_{[12][12]} - \Sigma_{[12][-12]}\Sigma_{[-12][-12]}^{-1}\Sigma_{[-12][12]}.$$

   The off diagonal entry of $(\Sigma^{-1})_{[12][12]}^{-1}$ is $\Sigma_{12} - \Sigma_{1[-12]}\Sigma_{[-12]}^{-1}\Sigma_{[-12]2} = \text{Cov}\,(e_1, e_2)$. Therefore, $(\Sigma^{-1})_{12} = 0$ is equivalent to $\text{Cov}\,(e_1, e_2) = 0$, i.e., X_1 and X_2 are partially uncorrelated. □

2. (1 point) (ANOVA for regression) Write the following ANOVA table of Y regressed on X in terms of projection matrices $P_X = X(X^TX)^{-1}X^T$, $P_1 = n^{-1}1_n1_n^T$.

Source	d.f.	SS	MSE	F
Model	p-1	SSM	MSM	MSM/MSE
Error	n-p	SSE	MSE	
Total	n-1	SST		

   For example, you can verify that SST $= Y^T(I - P_1)Y$

Solution:

$$\begin{aligned} SST = SSM + SSE &: \sum (y_i - \bar{y})^2 = \sum (\hat{y}_i - \bar{y})^2 + \sum (y_i - \hat{y}_i)^2 \\ Y = X\beta, P_X &= X(X^T X)^{-1} X^T, P_1 = n^{-1} 1_n 1_n^T, \\ SSM &= \sum_i (\hat{Y}_i - \bar{Y})^2 = Y^T (P_X - P_1) Y \\ SSE &= \sum_i (Y_i - \hat{Y}_i)^2 = Y^T (I - P_X) Y \\ SST &= \sum_i (Y_i - \bar{Y})^2 = Y^T (I - P_1) Y \end{aligned}$$

3. (1 point) X is a p-dimensional vector, taking values in R^p. $S = \sum_{i=1}^p |X_i|$. What is the integral Jacobian of $J(X \to S)$?

   Solution: Assume $X_1, \dots, X_p$ are iid standard Laplace random variables with PDF

$$f(x) = \frac{1}{2} e^{-|x|}.$$

   Therefore, the PDF of $\boldsymbol{X} = (X_1, \dots, X_p)^T$ is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = 2^{-p} e^{-\sum_{i=1}^p |x_i|} \equiv 2^{-p} e^{-s}.$$

   On the other hand, $S = \sum_{i=1}^p |X_i| \sim \text{Gamma}(p)$ with PDF

$$f_S(s) = \frac{1}{\Gamma(p)} s^{p-1} e^{-s} = \frac{1}{(p-1)!} s^{p-1} e^{-s}.$$

   Therefore, the integral Jacobian is

$$J(\boldsymbol{X} \to S) = \frac{f_S(s)}{f_{\boldsymbol{X}}(\boldsymbol{x})} = \frac{\frac{1}{(p-1)!} s^{p-1} e^{-s}}{2^{-p} e^{-s}} = \frac{2^p}{(p-1)!} s^{p-1}.$$

   □

4. (1 point) Show that the surface area of an n-dimensional unit sphere is $S_n = 2\pi^{n/2}/\Gamma(\frac{n}{2})$

   Adopt polar coordinate transform:

$$\begin{aligned} S_n &= \int_{\|\boldsymbol{x}\|=1} d\boldsymbol{x} = \int_{\rho=1} J(\boldsymbol{x} \to (\rho, \boldsymbol{\theta})) d\theta d\rho = J(\boldsymbol{x} \to \rho)|_{\rho=1} \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \end{aligned}$$

   □

   Remark: can also use geometric interpretation of integral Jacobian of polar transform $J(\boldsymbol{X} \to \rho)$.

5. (1 point) For the student score data (downloadable from the course web site), first standardize the columns so that each one has mean 0 and standard deviation 1.

   (a) Use singular value decomposition to compute $\hat{X}(1)$ and $\hat{X}(2)$. Graphically compare $\hat{X}(1)$ versus the original matrix X, and similarly $\hat{X}(2)$ versus X.

   (b) Compute $\|X - \hat{X}(J)\|^2$ for $J = 1, 2$, and verify the relationship $\|X - \hat{X}(J)\|^2 = \sum_{i=J+1}^k d_i^2$

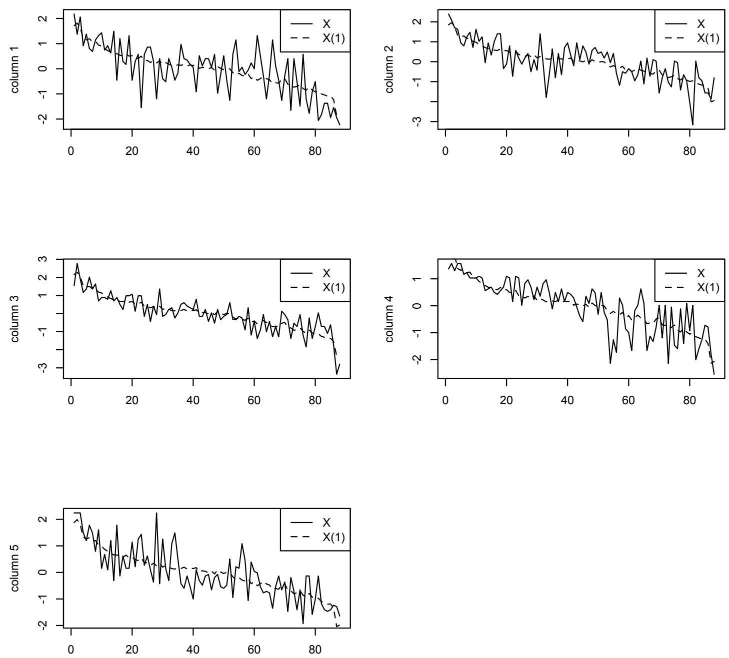
   Solution. (a) R code:

```
data_hw1   = read.table("scoredata.txt", header = FALSE)
for(k in 1:dim(data_hw1)[2]){
    temp.mean <- mean(data_hw1[,k])
    temp.sd <- sd(data_hw1[,k])
    data_hw1[,k] <- (data_hw1[,k] - temp.mean) / temp.sd
}

svd.hw1    = svd(data_hw1)
U          = svd.hw1$u
V          = svd.hw1$v
D          = svd.hw1$d

## X(1)
Xhat1      = D[1]*U[, 1]%*%t(V[, 1])
Xhat2      = Xhat1 + D[2]*U[, 2]%*%t(V[, 2])
```

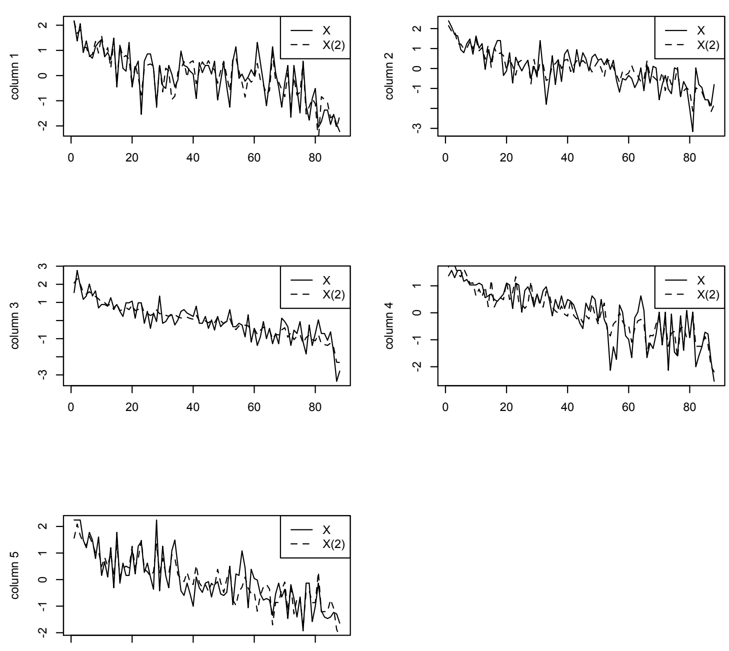
I plot five columns of $\widehat{X}(1)$ and $\widehat{X}(2)$ versus X in Figure 1.

(b) R code:

```
> c(sum((data_hw1 - Xhat1)^2), sum((D[2:5])^2))
[1] 158.2547  158.2547
> c(sum((data_hw1 - Xhat2)^2), sum((D[3:5])^2))
[1] 93.91198 93.91198
```



(a) $\widehat{X}(1)$ versus X: five columns



(b) $\widehat{X}(2)$ versus X: five columns

Figure 1: $\widehat{X}(J)$ versus X

6. (3 points) Let $Z \sim N_p(0, I)$ and $U = Z/\|Z\|$.

   (a) Use the polar coordinate to show that U is independent of $\|Z\|$.

   (b) How is density of U expressed in terms of the polar angles $(\theta_1, \theta_2, \ldots, \theta_{p-1})$?

Solution. (a) Define the polar transformation

$$\begin{cases} Z_1 &= \rho \cos\theta_1, \\ Z_2 &= \rho \sin\theta_1 \cos\theta_2, \\ Z_3 &= \rho \sin\theta_1 \sin\theta_2 \cos\theta_3, \\ \vdots & \\ Z_{p-1} &= \rho \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-2} \cos\theta_{p-1}, \\ Z_p &= \rho \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-2} \sin\theta_{p-1} \end{cases},$$

where $\rho = ||Z|| > 0$, $\theta_1, \ldots, \theta_{p-2} \in [0, \pi]$, and $\theta_{p-1} \in [0, 2\pi]$. The Jacobian of the transformation is

$$J(Z \to (\rho, \theta_1, \theta_2, \ldots, \theta_{p-1})) = \rho^{p-1} \sin^{p-2}\theta_1 \sin^{p-3}\theta_2 \cdots \sin\theta_{p-2}.$$

Therefore, the PDF of $(\rho, \theta_1, \theta_2, \ldots, \theta_{p-1})$ is

$$f(\rho, \theta_1, \theta_2, \ldots, \theta_{p-1}) \quad = \quad (2\pi)^{-p/2} \exp\left\{-\frac{\rho^2}{2}\right\} \rho^{p-1} \sin^{p-2}\theta_1 \sin^{p-3}\theta_2 \cdots \sin\theta_{p-2}$$

Since the PDF above can be factorized into two parts depending on ρ and $(\theta_1, \theta_2, \ldots, \theta_{p-1})$ respectively, we know that $\rho \perp\!\!\!\perp (\theta_1, \theta_2, \ldots, \theta_{p-1})$. Since $\rho = ||Z||$ and U is a function of $(\theta_1, \theta_2, \ldots, \theta_{p-1})$, we have $||Z|| \perp\!\!\!\perp U$.

(b) Since

$$\int_0^\infty \exp\left\{-\frac{\rho^2}{2}\right\} \rho^{p-1} d\rho = \int_0^\infty e^{-y} (2y)^{(p-1)/2} (2y)^{-1/2} dy = 2^{p/2-1} \Gamma\left(\frac{p}{2}\right),$$

the PDF of ρ is

$$f(\rho) = \frac{1}{2^{p/2-1}\Gamma\left(\frac{p}{2}\right)} \exp\left\{-\frac{\rho^2}{2}\right\} \rho^{p-1}.$$

And therefore the PDF of $(\theta_1, \theta_2, \ldots, \theta_{p-1})$ is

$$f(\theta_1, \theta_2, \ldots, \theta_{p-1}) = (2\pi)^{-p/2} 2^{p/2-1} \Gamma\left(\frac{p}{2}\right) \sin^{p-2}\theta_1 \sin^{p-3}\theta_2 \cdots \sin\theta_{p-2} = \frac{\Gamma\left(\frac{p}{2}\right)}{2\pi^{p/2}} \sin^{p-2}\theta_1 \sin^{p-3}\theta_2 \cdots \sin\theta_{p-2}.$$

The first $(p-1)$ components of U determines U_p up to a sign. When $U_p > 0$ or equivalently $\theta_{p-1} \in [0, \pi]$, the Jacobian from $(U_1, \ldots, U_{p-1})$ to $(\theta_1, \theta_2, \ldots, \theta_{p-1})$ is the absolute value of

$$\begin{aligned} \begin{vmatrix} \partial U_1/\partial\theta_1 & \cdots & \partial U_1/\partial\theta_{p-1} \\ \vdots & & \vdots \\ \partial U_{p-1}/\partial\theta_1 & \cdots & \partial U_{p-1}/\partial\theta_{p-1} \end{vmatrix} &= \begin{vmatrix} -\sin\theta_1 & 0 & \cdots & 0 \\ * & -\sin\theta_1 \sin\theta_2 & \cdots & 0 \\ \vdots & & & \vdots \\ * & * & * & -\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-1} \end{vmatrix} \\ &= (-1)^{p-1} \sin^{p-1}\theta_1 \sin^{p-2}\theta_2 \cdots \sin\theta_{p-1}. \end{aligned}$$

When $U_p < 0$ or equivalently $\theta_{p-1} \in [\pi, 2\pi]$, the Jacobian from $(U_1, \ldots, U_{p-1})$ to $(\theta_1, \theta_2, \ldots, \theta_{p-1})$ is the same as above. Therefore, the density of $(U_1, \cdots, U_{p-1}, U_p)$ expressed using $(\theta_1, \theta_2, \ldots, \theta_{p-1})$ is

$$\frac{\Gamma\left(\frac{p}{2}\right)}{2\pi^{p/2}} \times \frac{1}{\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-2} |\sin\theta_{p-1}|}.$$

□

7. (2 points) Let $X \sim N_p(0, \Sigma)$ be independent of $Z \sim N(0, 1)$. show that $Y = X/Z$ (multivariate Cauchy) has density function

$$f^Y(y) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\pi^{(p+1)/2}|\det(\Sigma)|^{1/2}}(1 + y^T\Sigma^{-1}y)^{-(p+1)/2}$$

and characteristic function $\phi_Y(t) = \exp\{-(t^T\Sigma t)^{1/2}\}$.

Solution. Define a one-to-one mapping from (X, Z) to $(Y = X/Z, Z)$ with Jacobian being the absolute value of the following determinant:

$$\left|\frac{\partial(Y, Z)}{\partial(X, Z)}\right| = \begin{vmatrix} 1/Z & 0 & 0 & \cdots & 0 & -X_1/Z^2 \\ 0 & 1/Z & 0 & \cdots & 0 & -X_2/Z^2 \\ 0 & 0 & 1/Z & \cdots & 0 & -X_3/Z^2 \\ \vdots & & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1/Z & -X_p/Z^2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = 1/Z^p.$$

Since the PDF of (X, Z) is

$$f_{X,Z}(x, z) = (2\pi)^{-p/2}|\det(\Sigma)|^{-1/2}\exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right)\cdot(2\pi)^{-1/2}\exp\left(-\frac{z^2}{2}\right),$$

the PDF of (Y, Z) is

$$f_{Y,Z}(y, z) = (2\pi)^{-p/2}|\det(\Sigma)|^{-1/2}\exp\left(-\frac{1}{2}z^2y^T\Sigma^{-1}y\right)\cdot(2\pi)^{-1/2}\exp\left(-\frac{z^2}{2}\right)|Z|^p.$$

The marginal PDF of Y is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{+\infty}(2\pi)^{-p/2}|\det(\Sigma)|^{-1/2}\exp\left(-\frac{1}{2}z^2y^T\Sigma^{-1}y\right)\cdot(2\pi)^{-1/2}\exp\left(-\frac{z^2}{2}\right)|z|^p dz \\ &= (2\pi)^{-p/2}|\det(\Sigma)|^{-1/2}(2\pi)^{-1/2}2\int_0^{+\infty} z^p\exp\left\{-\frac{z^2}{2}(1 + y^T\Sigma^{-1}y)\right\}dz. \end{aligned}$$

Let $c^2 = 1 + y^T\Sigma^{-1}y$, and we have

$$\begin{aligned} \int_0^{+\infty} z^p\exp\left\{-\frac{z^2}{2}c^2\right\}dz &= \int_0^{+\infty}\left(\frac{2w}{c^2}\right)^{p/2}e^{-w}\frac{(2w)^{-1/2}}{c}dw \\ &= 2^{(p-1)/2}c^{-(p+1)}\Gamma\left(\frac{p+1}{2}\right) \\ &= 2^{(p-1)/2}(1 + y^T\Sigma^{-1}y)^{-(p+1)/2}\Gamma\left(\frac{p+1}{2}\right). \end{aligned}$$

Therefore, the PDF of Y is

$$f_Y(y) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\pi^{(p+1)/2}|\det(\Sigma)|^{1/2}}(1 + y^T\Sigma^{-1}y)^{-(p+1)/2}.$$

The characteristic function of multivariate Cauchy is

$$\phi_Y(t) = E\left\{e^{it^TY}\right\} = E\left\{e^{it^TX/Z}\right\} = E\left\{e^{i\sqrt{t^T\Sigma t}Z'/Z}\right\},$$

since $t^TX \sim N(0, t^T\Sigma t) \sim \sqrt{t^T\Sigma t}Z'$ where Z' is another independent $N(0, 1)$. By the construction of standard univariate Cauchy Z'/Z and its characteristic function $E(e^{isZ'/Z}) = e^{-|s|}$, we immediately have

$$\phi_Y(t) = e^{-|\sqrt{t^T\Sigma t}|} = \exp\{-(t^T\Sigma t)^{1/2}\}.$$

□

Remark: an alternative way is to condition on Z first, then use Law of iterated expectation.