ISYE 7405

Homework 1

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1. Prove the following properties of matrix rank:

   (a) $rank(A_{p\times q}) + rank(B_{q\times r}) - q \leq rank(AB) \leq \min\{rank(A), rank(B)\}$, i.e. $rank(AB) = rank(A)$, if $rank(B) = q$

   Solution:

$$(1)\quad rank(A) + rank(B) = rank\begin{pmatrix} A & O \\ O & B \end{pmatrix} \leq rank\begin{pmatrix} A & O \\ I_q & B \end{pmatrix}$$

$$= rank\begin{pmatrix} A & O \\ I_q & B \end{pmatrix}\begin{pmatrix} I_q & -B \\ O & I_r \end{pmatrix}$$

$$= rank\begin{pmatrix} I_p & -A \\ O & I_q \end{pmatrix}\begin{pmatrix} A & -AB \\ I_q & O \end{pmatrix}$$

$$= rank\begin{pmatrix} O & -AB \\ I_q & O \end{pmatrix} = rank(AB) + q$$

   (2) $L_{col}(AB) \subset L_{col}(A), \quad L_{row}(AB) \subset L_{row}(B)$

   (b) $rank(AB) + rank(BC) \leq rank(B) + rank(ABC)$

   Solution:

$$rank(AB) + rank(BC) = rank\begin{pmatrix} AB & O \\ O & BC \end{pmatrix} \leq rank\begin{pmatrix} AB & O \\ B & BC \end{pmatrix}$$

$$= rank\begin{pmatrix} AB & O \\ B & BC \end{pmatrix}\begin{pmatrix} I & -C \\ O & I \end{pmatrix}$$

$$= rank\begin{pmatrix} I & -A \\ O & I \end{pmatrix}\begin{pmatrix} AB & -ABC \\ B & O \end{pmatrix}$$
$$= rank\begin{pmatrix} O & B \\ ABC & O \end{pmatrix}$$
$$= rank(B) + rank(ABC)$$

(c) $rank(A + B) \leq rank(A) + rank(B)$

Solution:

$L_{col}(A + B) \subset Span\{L_{col}(A) \cup L_{col}(B)\}$

2. Prove the following properties of matrix trace:

(a) $\text{tr}(AB) = \text{tr}(BA)$

Solution:

$\text{tr}(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j a_{ij}a_{ji} = \sum_j \sum_i b_{ji}a_{ij} = \sum_j (BA)_{ji} = \text{tr}(BA)$.

(b) $\text{tr}(A) = \sum \lambda_i$, where λ_i are eigenvalues of A, if A is real symmetric.

Solution:

Since A is symmetric, it can be decomposed as $A = P\Lambda P^T$, where P is an orthogonal matrix satisfying $PP^T = P^TP = I$ and $\Lambda = \text{diag}(\lambda_i)$. Then we have $\text{tr}(A) = \text{tr}(P\Lambda P^T) = \text{tr}(P^TP\Lambda) = \text{tr}(\Lambda) = \sum \lambda_i$.

3. Let A be a $n \times n$ such that $A^TA = A^2$. Show that A is symmetric.

Solution:

$tr[(A - A^T)^T(A - A^T)] = tr(A^TA - AA - AA^T + AA^T) = 0$.

4. Calculate the Vandermonde determinant.

$$V(x_1,...,x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{n-1} \end{vmatrix}$$

Solution:

Starting from the left column, we minus x_1 multiply the column left to it.

$$\begin{aligned} V(x_1,...,x_n) &= \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{n-1} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_1 x_2 & \cdots & x_2^{n-2} - x_1 x_2^{n-3} & x_2^{n-1} - x_1 x_2^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n - x_1 & x_n^2 - x_1 x_n & \cdots & x_n^{n-2} - x_1 x_n^{n-3} & x_n^{n-1} - x_1 x_n^{n-2} \end{vmatrix} \\ &= \begin{vmatrix} x_2 - x_1 & x_2^2 - x_1 x_2 & \cdots & x_2^{n-2} - x_1 x_2^{n-3} & x_2^{n-1} - x_1 x_2^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n - x_1 & x_n^2 - x_1 x_n & \cdots & x_n^{n-2} - x_1 x_n^{n-3} & x_n^{n-1} - x_1 x_n^{n-2} \end{vmatrix} \\ &= (x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1) \begin{vmatrix} 1 & x_2 & \cdots & x_2^{n-3} & x_2^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-3} & x_n^{n-2} \end{vmatrix} \\ &= \prod_{j>1} (x_j - x_1) V(x_2,...,x_n) \\ &= \prod_{1 \le i < j \le n} (x_i - x_j) \end{aligned}$$

5. (Sherman-Morrison formula) Suppose the inverse of $A = A_{n\times n}$ is known, now we perturb each element of the matrix by $A_{ij} \to A_{ij} + \Delta x_i \Delta y_j = B_{ij}$, denote the new matrix by $B = \{B_{ij}\}$. Calculate the inverse of matrix B.

   Solution:

   Write $B = A + \boldsymbol{u}\boldsymbol{v}^T$, where $\boldsymbol{u}$, $\boldsymbol{v}$ are $n \times 1$ vectors where $u_i = \Delta x_i, v_i = \Delta y_i$. The Sherman-Morrison formula provides a quick way to compute the inverse of matrix B:

$$B^{-1} = (A + \boldsymbol{u}\boldsymbol{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\boldsymbol{u}\boldsymbol{v}^T A^{-1}}{1 + \boldsymbol{v}^T A^{-1}\boldsymbol{u}}$$

6. Consider the equicorrelation matrix

$$A = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}$$

   Find the explicit expressions of $\det(A)$ and A^{-1}.

   Solution:

   We can write A as follows:

$$A = (1-\rho)I_n + \rho \mathbf{1}_n\mathbf{1}_n' = (1-\rho)(I_n + \frac{\rho}{1-\rho}\mathbf{1}_n\mathbf{1}_n').$$

   Therefore, the determinant of A is

$$\det(A) = (1-\rho)^n \det(I_n + \frac{\rho}{1-\rho}\mathbf{1}_n\mathbf{1}_n') = (1-\rho)^n \det(1 + \frac{\rho}{1-\rho}\mathbf{1}_n'\mathbf{1}_n) = (1-\rho)^n(1 + \frac{n\rho}{1-\rho}).$$

   The inverse of A is:

$$\begin{aligned} A^{-1} &= (1-\rho)^{-1}\left\{ I_n^{-1} - I_n^{-1}\mathbf{1}_n\mathbf{1}_n' I_n^{-1} / \left(\frac{1-\rho}{\rho} + \mathbf{1}_n' I_n^{-1}\mathbf{1}_n \right) \right\} \\ &= \frac{1}{1-\rho} I_n - \frac{\rho}{(1-\rho)^2 + \rho(1-\rho)n}\mathbf{1}_n\mathbf{1}_n'. \end{aligned}$$