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Combinatorial Convexity and Algebraic Geometry

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1 Introduction and Preliminaries

Convex Bodies

Convex sets

For most of my notes, the sets we'll be considering are subsets of Euclidean $n-$space. Many definitions and theorems could be stated in an affinely invariant manner. I won't, however, stress this point. If we're using the symbol $\mathbb{R}^n$, it should be clear from the context whether we mean real vector space, real affine space, or Euclidean space. In the latter case, we assume the ordinary scalar product

$$\langle x, y\rangle = \xi_1\eta_1 + \ldots + \xi_n\eta_n \qquad \text{for } x = (\xi_1, \ldots, \xi_n),\ y = (\eta_1, \ldots, \eta_n)$$

so that the square of Euclidean distance between points x and y equals

$$\|x - y\|^2 = \langle x - y, x - y\rangle.$$

Recall that an open ball with center x and radius r is the set $\{y \mid \|x - y\| < r\}$. By $\langle K, y\rangle \geq 0$, we mean $\langle x, y\rangle \geq 0$ for every $x \in K$. We assume the reader to be somewhat familiar with $n-$dimensional affine and Euclidean geometry.

1.1 Definition. A set $C \subset \mathbb{R}^n$ is called *convex* if, for all $x, y \in C, x \neq y$, the line segment

$$[x, y] := \{\lambda x + (1 - \lambda)y \mid 0 \leq \lambda \leq 1\}$$

is contained in C (Figure 1).

Examples of convex sets are a point, a line, a circular disc in $\mathbb{R}^2$, the platonic solids (see Figure 10 in section 6) in $\mathbb{R}^3$. Also $\emptyset$ and $\mathbb{R}^n$ are convex.

If B is an open circular disc in $\mathbb{R}^2$ and M is any subset of the boundary circle ∂B of B, then $B \cup M$ is also convex. So, a convex set need be neither open nor closed. In general we shall restrict ourselves to closed convex sets.

There is a simple way to construct new convex sets from given ones:

1.2 Lemma. *The intersection of an arbitrary collection of convex sets is convex.*

Proof: If a line segment is contained in every set of the collection, it is also contained in their intersection.

□

1.3 Definition. We say x is a convex combination of $x_1, \ldots, x_r \in \mathbb{R}^n$ if there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ such that

$$x = \lambda_1 x_1 + \ldots + \lambda_r x_r \,, \tag{1}$$

$$\lambda_1 + \ldots + \lambda_r = 1 \,, \tag{2}$$

$$\lambda_1 \geq 0 \,, \ldots \,, \lambda_r \geq 0. \tag{3}$$

If condition (3) is dropped, we have an *affine* combination of $x_1, \ldots, x_r$, and $x, x_1, \ldots, x_r$ are called *affinely dependent.* If $x, x_1, \ldots, x_r$ are not affinely dependent, we say they are *affinely independent.*

So, convex combinations are special affine combinations (Figure 2).

If $x_1, \ldots, x_r$ are affinely independent, the numbers $\lambda_1, \ldots, \lambda_r$ are sometimes called barycentric coordinates of x (with respect to the *affine basis* $x_1, \ldots, x_r$).

1.4 Definition. The set of all convex combinations of a set $M \subset \mathbb{R}^n$ is called the *convex hull*

$$\text{conv M}$$

of M; in particular, conv $\emptyset = \emptyset$. Analogously, the set of all affine combinations of elements of M is called the *affine hull*

$$\text{aff M}$$

of M. We will denote by lin M (*linear hull)* the linear space generated by M. It is the "smallest" linear space containing M.

If $M = \{x_1, \ldots, x_r\}$ is a finite set, we say $P :=$ conv M is a *convex polytope*, or simply a *polytope.*

If $x_1, \ldots, x_r$ are affinely independent, we say

$$T_{r-1} := \text{ conv}\{x_1, \ldots, x_r\}$$

is an $(r-1)-$simplex or, briefly, a *simplex.* aff T_{r-1} and T_{r-1} are said to have *dimension* $r-1$.

Remarks.

1. Clearly, $M \subset$ conv $M \subset$ aff M.

2. Every polytope is compact (that is, bounded and closed).

1.5 Theorem.

1. *(a) A set $M \subset \mathbb{R}^n$ is convex if and only if it contains all its convex combinations, that is, if and only if*

$$M = \text{conv } M$$

2. *The convex hull of $M \subset \mathbb{R}^n$ is the smallest convex set that contains M; this means $M \subset M'$ and M' convex imply conv $M \subset M'$*

Proof. First, we will show that conv M is convex.

If $x, y \in$conv M, there exist $x_1, \ldots, x_r, y_1, \ldots, y_s \in M$ and real numbers $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s$ such that

$$x = \lambda_1 x_1 + \ldots + \lambda_r x_r, \quad \lambda_1 + \ldots + \lambda_r = 1 \quad \lambda_1 \geq 0, \ldots, \lambda_r \geq 0$$

and

$$y = \lambda_1 y_1 + \ldots + \lambda_s y_s, \quad \lambda_1 + \ldots + \lambda_s = 1 \quad \lambda_1 \geq 0, \ldots, \lambda_s \geq 0.$$

Employing 0 coefficients, if necessary, we may assume $r = s$ and $y_j = x_j, j = 1, \ldots, r$. For arbitrary $0 \leq \lambda \leq 1$,

$$\begin{aligned} &\lambda x + (1-\lambda) y = \lambda(\lambda_1 x_1 + \ldots + \lambda_r x_r) + (1-\lambda)(\mu_1 x_1 + \ldots + \mu_r x_r) \\ &= [\lambda\lambda_1 + (1-\lambda)\mu_1]x_1 + \ldots + [\lambda\lambda_r + (1-\lambda)\mu_r]x_r. \end{aligned}$$

Since all coefficients are nonnegative, and since

$$\lambda\lambda_1 + (1-\lambda)\mu_1 + \ldots + \lambda\lambda_r + (1-\lambda)\mu_r = \lambda + 1 - \lambda = 1\,,$$

$\lambda x + (1-\lambda) y$ is a convex combination of $x_1, \ldots, x_r$. So, conv M is convex and, in view of Remark 1, we obtain (a).

Now, to see (b), suppose M' is a convex set, $M' \supset M$, and that $x \in$ conv M. Then there exist $x_1, \ldots, x_r \in M$ such that $x = \lambda_1 x_1 + \ldots + \lambda_r x_r, \lambda_1 + \ldots + \lambda_r = 1$, and $\lambda_1, \ldots, \lambda_r > 0$. Since $x_1, \ldots, x_r \in M'$ as well, we find successively

$$\begin{aligned} &y_1 := \lambda_1(\lambda_1+\lambda_2)^{-1}x_1 + \lambda_2(\lambda_1+\lambda_2)^{-1}x_2 \\ &y_2 := (\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_3)^{-1}y_1 + \lambda_3(\lambda_1+\lambda_2+\lambda_3)^{-1}x_3 \\ &\vdots x = (\lambda_1 + \ldots + \lambda_{r-1})(\lambda_1 + \ldots + \lambda_r)^{-1}y_{r-2} + \lambda_r(\lambda_1 + \ldots + \lambda_r)^{-1}x_r \end{aligned}$$

which are all in M', hence, conv $M \subset M'$.

□

1.6 Definition. If C is a convex set, we call

$$\dim C := \dim (\operatorname{aff} C)$$

the *dimension* of C By convention, dim $\emptyset = -1$.

1.7 Definition. A compact convex set C is called a *convex body.*

For example, note that points and line segments are convex bodies in $\mathbb{R}^n, n \geq 1$, so that a convex body in $\mathbb{R}^n$ need not have dimension n.

1.8 Definition. We say $x \in M \subset \mathbb{R}^n$ is in the *relative interior* of M, $x \in \text{rint } M$, if x is in the interior of M relative to aff M (that is, there exists an open ball B in aff M such that $x \in B \subset M$). If aff $M = \mathbb{R}^n$, then rint $M :=$ int M(note that rint $\mathbb{R}^0 = \text{int } \mathbb{R}^0 = \{0\}$).

Our main emphasis will be on convex polytopes and an unbounded counterpart of polytopes, called polyhedral cones:

1.9 Definition. If $M \subset \mathbb{R}^n$, the set of all nonnegative linear combinations

$$x = \lambda_1 y_1 + \ldots + \lambda_k y_k\,, \quad y_1, \ldots, y_k \in M\,, \quad \lambda_1 \geq 0, \ldots, \lambda_k \geq 0$$

of elements of M is called the *positive hull*

$$\sigma := \text{ pos } M$$

of M or the *cone* determined by M. By convention, pos $\emptyset := \{0\}$.

For fixed $u \in \mathbb{R}^n, u \neq 0$, and $\alpha \in \mathbb{R}$, the set $H := \{x \mid \langle x, u\rangle = \alpha\}$ is a hyperplane. $H^+ := \{x \mid \langle x, u\rangle \geq \alpha\}$ and $H^- := \{x \mid \langle x, u\rangle \leq \alpha\}$ are called the *half-spaces bounded by* H. If $\sigma \subset H^+$ and $\alpha = 0$, we say σ has an *apex*, namely 0. (we use the symbol 0 for the number 0, the zero vector, and the origin).

If $M = \{x_1, \ldots, x_r\}$ is finite, we call

$$\sigma = \text{ pos } \{x_1, \ldots, x_r\}$$

a *polyhedral cone.* Unless otherwise stated, by a *cone* we always mean a polyhedral cone. Sometimes we write

$$\sigma = \mathbb{R}_{\geq 0} x_1 + \ldots + \mathbb{R}_{\geq 0} x_r\,,$$

$\mathbb{R}_{\geq 0}$ denoting the set of nonnegative real numbers. From now on, we will use the notation $\mathbb{R}_+$ denoting this set of *nonnegative* real numbers and $\mathbb{R}_{++}$ denoting the set of *strictly positive* real numbers.

Example. A quadrant in $\mathbb{R}^2$ and an octant in $\mathbb{R}^3$ are cones with an apex, whereas a closed half-space or the intersection of two closed half-spaces H_1^+, H_2^+ with $0 \in H_1, 0 \in H_2$ in $\mathbb{R}^3$, are cones without apex.

Since convex combinations are, by definition, nonnegative linear combinations, we have

1.10 Lemma. *The positive hull of any set M is convex.*

Figure 3 illustrates a polyhedral cone of dimension three which is the positive hull of two-dimensional polytope K. Through pos M might generally be called a cone, we reserve this term for polyhedral cones.

Section Exercises

1. The convex hull of any compact (closed and bounded) set is again compact.

2. Find an example of a closed set M such that conv M is not closed.

3. Determine all convex subsets C of $\mathbb{R}^3$, for which $\mathbb{R}^3 \backslash C$ is also convex. (Except $\emptyset, \mathbb{R}^{\not\models}$ there are, up to three such sets of affine transformations, that is, translations combined with linear maps.

4. Call a set M $\epsilon-$convex if, for a given $\epsilon > 0$, each ball with radius ϵ and center in M intersects M in a convex set. Furthermore, call a set M connected if any two if its points can be joined by a rectifiable arc (as is defined in calculus) contained in M. Prove: (a) Any $\epsilon-$convex closed connected set M in $\mathbb{R}^2$ is convex. (b) Statement (a) is false without the assumption of M being connected.

Theorems of Radon and Caratheodory

The following theorem is helpful when handling convex combinations.

2.1 Theorem (Radon's Theorem). *Let $M = \{x_1, \ldots, x_r\} \subset \mathbb{R}^n$ be an arbitrary finite set, and let M_1, M_2 be a partition of M, that is, $M = M_1 \cup M_2, M_1 \cap M_2 = \emptyset$, $M_1 \neq \emptyset, M_2 \neq \emptyset$.*

1. (a) *If $r \geq n+2$ then the partition can be chosen such that*

$$\text{conv } M_1 \cap \text{ conv } M_2 \neq \emptyset.$$

2. (b) *If $r \geq n+1$ and 0 is an apex of pos M, yet $0 \notin M$ or $r \geq n+2$, then the partition can be chosen such that*

$$\text{pos } M_1 \cap \text{ pos } M_2 \neq \{0\}.$$

3. (c) *The partition is unique if and only if, in case (a), $r = n+2$ and any $n+1$ points of M are affinely independent, in case (b), $r = n+1$ and any n points of M are linearly independent.*

2.2 Definition. We call M_1, M_2 in **Theorem 2.1** a *Radon partition* of M.

Proof of Theorem 2.1

(a) From $r \geq n+2$, it follows that $x_1, \ldots, x_r$ are affinely dependent. Hence,

$$\lambda_1 x_1 + \ldots + \lambda_r x_r = 0 \text{ can hold with } \lambda_1 + \ldots + \lambda_r = 0\,,\ \text{not all } \lambda_i = 0.$$

We may assume that, for a particular $j, 0 < j < r$,

$$\lambda_1 > 0, \ldots, \lambda_j > 0; \quad \lambda_{j+1} \leq 0, \ldots, \lambda_r \leq 0$$

We set

$$\begin{aligned} &\lambda := \lambda_1 + \ldots + \lambda_j = -\lambda_{j+1} - \ldots - \lambda_r > 0 \quad \text{ and} \\ &x := \lambda^{-1}(\lambda_1 x_1 + \ldots + \lambda_j x_j) = -\lambda^{-1}(\lambda_{j+1} x_{j+1} + \ldots + \lambda_r x_r). \end{aligned}$$

Then, $x \in \text{conv } M_1 \cap \text{conv } M_2$ for

$$M_1 := \{x_1, \ldots, x_j\}\,, \quad M_2 := \{x_{j+1}, \ldots, x_r\}.$$

(b) We prove the uniqueness only in case (a); case (b) is proved similarly. First, assume $r = n+2$ and no $n+1$ points are affinely dependent. Suppose that

$$\tilde{M}_1 = \{x_1, \ldots, x_{i_k}\}, \quad \tilde{M}_2 = \{x_{i_{k+1}}, \ldots, x_{i_{n+2}}\}$$

is a second Radon partition of M and

$$y \in \text{ conv } \tilde{M}_1 \cap \text{ conv } \tilde{M}_2.$$

Then,

$$y = \mu^{-1}(\mu_1 x_{i_1} + \ldots + \mu_k x_{i_k} = -\mu^{-1}(\mu_{k+1} x_{i_{k+1}} + \ldots + \mu_{n+2} x_{i_{n+2}})$$

where $\mu_1 > 0, \ldots, \mu_k > 0; \mu_{k+1} \leq 0, \ldots, \mu_{n+2} \leq 0; k \geq 1$, and $\mu = \mu_1 + \ldots + \mu_k = -\mu_{k+1} - \ldots - \mu_{n+2}$. We may assume

$$x_{i_1} = x_{j+1} \quad (\in M_2)$$

We choose $0 < \alpha < 1$ such that

$$\alpha\lambda^{-1}\lambda_{j+1} + (1-\alpha)\mu^{-1}\mu_1 = 0$$

Then,

$$\begin{aligned} &\alpha\lambda^{-1}(\lambda_1 x_1 + \ldots + \lambda_{n+2} x_{n+2}) \\ &+ (1-\alpha)\mu^{-1}(\mu_1 x_1 + \ldots + \mu_{n+2} x_{i_{n+2}}) = 0 + 0 = 0 \end{aligned}$$

and

$$\alpha\lambda^{-1}(\lambda_1 + \ldots + \lambda_{n+2}) + (1-\alpha)\mu^{-1}(\mu_1 + \ldots + \mu_{n+2}) = 0.$$

expresses an affine relation between $n+1$ of the points of M (x_{i_1} and x_{j+1} cancel out), unless all coefficients vanish. Therefore, $\lambda_\varphi = -\alpha^{-1}(1-\alpha)\lambda\mu^{-1}\mu_{i_\varphi}\,, \varphi = 1, \ldots, n+2$, and there is a map $\varphi \mapsto \varphi', \varphi \in \{1, \ldots, j, j+2, \ldots, n+2\}, \varphi' \in \{i_2, \ldots, n+2\}$ such that $\lambda_\varphi = -\alpha^{-1}(1-\alpha)\lambda\mu_{\varphi'}$. Since $\alpha^{-1} > 0, 1-\alpha > 0$, and $\lambda > 0$, the set of those φ' for which $\mu_{\varphi'} < 0$ is the same as the set of those φ for which $\lambda_\varphi > 0$. Therefore $M_1 = \{x_1, \ldots, x_j\} = \{x_{i_{k+1}}, \ldots, x_{i_{n+2}}\} = \tilde{M}_2$ and consequently $M_2 = \tilde{M}_1$,too.

To prove the converse, we distinguish these two cases.

(I) $r = n+2$, and $x_1, \ldots, x_{n+1}$ are affinely dependent, $\overset{\circ}{M} := \{x_1, \ldots, x_{n+1}\}$.

(II) $r > n+2$.

In case I, $\overset{\circ}{M}$ is contained in a hyperplane so that, by (a), we find a partition of $\overset{\circ}{M}$ into $\overset{\circ}{M}_1, \overset{\circ}{M}_2$ with conv $\overset{\circ}{M}_1 \cap$ conv $\overset{\circ}{M}_2 \neq \emptyset$. Then, $\overset{\circ}{M}_1 \cup \{x_{n+2}\}$, $\overset{\circ}{M}_2$ and $\overset{\circ}{M}_1$, $\overset{\circ}{M}_2 \cup \{x_{n+2}\}$ are two different Radon partitions of M.

In case II, consider a proper subset $\tilde{M}$ of M which has at least $n+2$ points. Let $\tilde{M}_1, \tilde{M}_2$ be a Radon partition of $\tilde{M}$. Then, $\tilde{M}_1 \cup (M \backslash \tilde{M}), \tilde{M}_2$ and $\tilde{M}_1,\ \tilde{M}_2 \cup (M \backslash \tilde{M})$ are different Radon partitions of M.

□

2.3 Theorem (Caratheodory's theorem).
(a) *The convex hull conv M of a set $M \subset \mathbb{R}^n$ is the union of all convex hulls of subsets of M containing at most $n+1$ elements.*
(b) *The positive hull pos M of a set $M \subset \mathbb{R}^n$ is the union of all positive hulls of subsets of M containing at most n elements of M.*

Proof:
(a) Let

$$x = \lambda_1 x_1 + \ldots + \lambda_r x_r \in \text{ conv M},$$

and let r be the smallest number of elements of M of which x is a convex combination. Contrary to the claim, $r \geq n+2$ implies there exists an affine relation

$$\mu_1 x_1 + \ldots + \mu_r x_r = 0, \text{ with } \mu_1 + \ldots + \mu_r = 0, \quad \text{ but not all } \mu_j = 0.$$

For $\mu_j \neq 0$, we obtain from (1) and (2)

$$x = \lambda_1 x_1 + \ldots + \lambda_r x_r = \left(\lambda_1 - \frac{\lambda_j}{\mu_j}\mu_1\right)x_1 + \ldots + \left(\lambda_r - \frac{\lambda_j}{\mu_j}\mu_r\right)x_r.$$

We may assume $\mu_j > 0$, and, for all $\mu_k > 0, k = 1, \ldots, r$,

$$\frac{\lambda_j}{\mu_j} \leq \frac{\lambda_k}{\mu_k}.$$

Then,

$$\lambda_i - \frac{\lambda_j}{\mu_j}\mu_i \geq 0 \qquad \text{for } i = 1, \ldots, r.$$

Since $\lambda_j - \frac{\lambda_j}{\mu_j}\mu_j = 0$, equation (3) expresses x as a convex combination of less than r elements of M, a contradiction of the initial assumption.

(b) Replace in the proof of (a) "convex combination" by "positive linear combination" and "affine dependence of $n+1$ elements" by "linear dependence of n elements" to obtain a proof of (b).

□

Exercises

1. In analogy to the above examples in Figure 4, find all types of Radon partitions of $n+2$ points in $\mathbb{R}^n$ whose affine hull is $\mathbb{R}^n$.

2. If aff $M = \mathbb{R}^n$, then, conv M is the union of $n-$simplices with vertices in M.

3. Every $n-$dimensional convex polytope is the union of finitely many simplices, no two of which have an interior point in common.

4. Helly's Theorem. Suppose every $n+1$ of the convex sets $K_1, \ldots, K_m$ in $\mathbb{R}^n$ has a nonempty intersection, $m \geq n+1$. Then $\bigcap_{i=1}^m K_i \neq \emptyset$. (Hint: For $m = n+1$ there is nothing to prove. Apply induction on m and use Radon's Theorem).

Nearest point map and supporting hyperplanes

Quite a few properties of a closed convex set K can be studied by using the map that assigns to each point in $\mathbb{R}^n$ its nearest point on K. First, we show that this map is well defined.
3.1 Lemma. *Let K be a closed convex set in $\mathbb{R}^n$. To each $x \in \mathbb{R}^n$ there exists a unique $x' \in K$ such that*

$$\|x - x'\| = \inf_{y \in K} \|x - y\|. \tag{*}$$

Proof: The existence of an x' satisfying $(*)$ follows from K being closed. Suppose that, for $x'' \in K$, $x'' \neq x'$,

$$\|x - x''\| = \inf_{y \in K} \|x - y\|.$$

Consider the isosceles triangle with vertices x, x', x''. The midpoint $m = \frac{1}{2}(x' + x'')$ of the line segment between x' and x'' is, by convexity, also in K, but satisfies

$$\|x - m\| < \inf_{y \in K} \|x - y\| ,$$

a contradiction.

□

3.2 Definition. The map

$$\begin{aligned} p_K &: \mathbb{R}^n \to K \\ x &\mapsto p_K(x) = x' \end{aligned}$$

of lemma 3.1 is called the *nearest point map* relative to K.
Clearly,
3.3 Lemma.

1. (a) $p_K(x) = x$ *if and only if* $x \in K$;

2. (b) p_K is surjective.

Generalizing the concept of a *tangent hyperplane* is the following.

3.4 Definition. A hyperplane H is called a *supporting hyperplane* of a closed convex set $K \subset \mathbb{R}^n$ if $K \cap H \neq \emptyset$ and $K \subset H^-$ or $K \subset H^+$.

We call H^- (or H^+, respectively) a *supporting half-space* of K (possibly $K \subset H$).

If u is a normal vector of H pointing into H^+(or H^-, respectively), we say that u is an *outer normal* of K (Figure 5), and $-u$ an *inner normal* of K.

3.5 Lemma. *Let $\emptyset \neq K \subset \mathbb{R}^n$ be closed and convex. For every $x \in \mathbb{R}^n \backslash K$ the hyperplane H containing $x' := p_K(x)$ and perpendicular to the line joining x and x' is a supporting hyperplane of K described by $H = \{y \mid \langle y, u\rangle = 1\}$, for $u = \frac{x-x'}{\langle x', x-x'\rangle}$, unless H contains* 0.
Proof: The hyperplane $H := \{y \mid \langle y, u\rangle = 1\}$ (u as before) is perpendicular to $x - x'$ and satisfies $x' \in H$. Moreover, $\langle x - x', x - x'\rangle > 0$ implies $\langle x, x - x'\rangle > \langle x', x - x'\rangle$ and, thus, $x \in H^+$. Suppose H is not a supporting hyperplane of K. Then there exists some $y \in K \cap (H^+ \backslash H), y \neq x$. By elementary geometry applied to the plane E spanned by x, x', and y, the line segment $[y, x']$ contains a point z interior to the circle in E about x with radius $\|x - x'\|$. Then, $\|x - z\| < \|x - x'\|$, a contradiction.

3.6 Lemma. *Let $K \subset \mathbb{R}^n$ be closed and convex, and let $x \in \mathbb{R}^n \backslash K$. Suppose y lies on the ray emanating from x' and containing x. Then $x' = y'$.*
Proof: First, assume $y \in [x, x']$. Then in the case $x' \neq y'$,

$$\|x - x'\| = \|y - x'\| + \|x - y\| > \|y - y'\| + \|x - y\| \geq \|x - y'\| ,$$

a contradiction.

If $x \in [y, x'], x' \neq y'$, then, the line parallel to $[y, x']$ through x meets $[x', y']$ in a point $x_0 \neq x'$. From $\|x - x_0\| = \|x - x'\| \frac{\|y-y'\|}{\|y-x'\|}$ (similar triangles) and $\|y - y'\| < \|y - x'\|$ (Lemma 3.1), we obtain $\|x - x_0\| < \|x - x'\|$, a contradiction.

□

3.7 Lemma (Busemann and Feller's lemma). *p_K does not increase distances, and, hence, is Lipschitz with Lipschitz constant* 1. *In particular, p_K is uniformly continuous.*

Proof: Let $x, y \in \mathbb{R}^n \backslash K$. For $p_K(x) = p_K(y)$, the lemma is trivial; so, suppose $p_K(x) \neq p_K(y)$, and let g be the line through $x' := p_K(x)$ and $y' := p_K(y)$. We denote by H_1, H_2 the hyperplanes perpendicular to g in x', y', respectively.

Neither of x and y lies in the open stripe S bounded by H_1 and H_2, for if, say, x does, the foot x_0(orthogonal projection) of x on g lies in K, and then

$$\|x - x_0\| < \|x - x'\| ,$$

a contradiction. Also, the points x, y *cannot lie* on the same side of H_1 or H_2 opposite to S since $[x, x'] \cap (S \backslash K) \neq \emptyset$ or $[y, y'] \cap (S \backslash K) \neq \emptyset$ would contradict what we have just shown and Lemma 3.6.

3.8 Theorem. *A closed convex proper subset of $\mathbb{R}^n$ is the intersection of its supporting half-spaces.*

Proof: By Lemma 3.5, there exists a supporting half-space of K. Let $K' := \bigcap H^+$ for all supporting half-spaces H^+ of K. Clearly, $K \subset K'$.

Suppose $x \in K' \backslash K$. Then $p_K(x) \neq x$ and, hence, by Lemma 3.5, the hyperplane perpendicular in $p_K(x)$ to the line joining x and $p_K(x)$ separates x and K, so that $x \notin K'$, a contradiction.

□

Remark. In general, not all supporting half-spaces of K are needed to represent K as their intersection. A triangle in $\mathbb{R}^2$, for example, has infinitely many supporting half-places, but three half-planes already suffice to represent the triangle as their intersection.

3.9 Theorem. *Any closed convex set K possesses a supporting hyperplane at each of its boundary points.*

Proof: Suppose $x_0 \in \partial K$ is a boundary point of K, that is, any open disc U_δ with center x_0 and radius $\delta > 0$ contains points from $\mathbb{R}^n \backslash K$. Then, x_0 is the limit point of a sequence $\{x_j\} \to x_0$ with $x_j \in \partial K$, such that there exist supporting hyperplanes H_i of K at x_i according to Lemma 3.5. Let s_i be the ray of outer normals of H_i in $i = 1, 2, \ldots$, and let S be a sphere with center x_0.

For sufficiently large $i, s_i \cap S$ is a point y_i, and $x_i = p_K(y_i)$ by Lemma 3.6. $\{y_i\}$ has a cluster point $y_0 \neq x_0$. Since p_K is a continuous (Lemma 3.7), $p_K(y_0) = x_0$ and $y_0 \notin K$ otherwise $p_K(y_0) = y_0 = x_0$ would follow. Therefore, Lemma 3.5 applies, and the theorem follows.

□

Exercises

1. Let $K \subset \mathbb{R}^n$ be closed and convex. Then, dim $K = k$ if and only if, for any $x \in$rint K, the set $p_K^{-1}(x)$ is an $(n-k)-$dimensional affine space, $0 \leq k \leq n$.
2. Every closed convex set is the intersection of countably many of its supporting half-spaces.
3. Let $M \subset \mathbb{R}^n$ be compact. pos M has an apex if $0 \notin$ conv M.
4. A closed set $K \subset \mathbb{R}^n$ that possesses a well-defined nearest point map is convex. (Hint: Reduce the problem to $n = 2$. Use increasing sequences $B_1 \subset B_2 \subset \ldots$ of circular discs $B_j \subset \mathbb{R}^2 \backslash K\,, j = 1, 2, \ldots$.

Faces and normal cones

Although faces and normal cones will mainly be used in the special case of polytopes, we introduce them for closed convex sets. This lets us see properties specific to polytopes.

4.1 Definition. If H is a supporting hyperplane of the closed convex set K, we call $F := K \cap H$ a *face* of K. By convention, $\emptyset$ and K are called *improper faces* of K.

If we speak about faces, it should be clear from the context whether we include $\emptyset$ or K or not.

By Lemma 1.2,

4.2 Lemma *Every face of a closed convex set K is again a closed convex set.*

So we can speak about the dimension of a face. Recall the convention dim $\emptyset = -1$.

4.3 Definition. By a $k-$face F of K, we mean a face of dimension k. We call F

- (a) a *vertex* of K, if $k = 0$
- (b) an *edge* of K, if $k = 1$,
- (c) a *facet* of K, if $k = \dim K - 1$.

We denote the set of vertices of K by vert K.

4.4 Lemma. *Let F_0 and F_1 be faces of a closed convex set K such that $F_0 \subset F_1$. Then, F_0 is a (possibly improper) face of F_1.*

Proof: Let $F_0 = K \cap H_0$, where H_0 is a supporting hyperplane of K and, hence, also of F_1. Then,

$$F_1 \cap H_0 \subset K \cap H_0 = F_0 \subset F_1 \cap H_0 \,,$$

hence, $F_0 = F_1 \cap H_0$ which proves the lemma.

□

Remark. The converse of Lemma 4.4 is false. As Figure 6 illustrates, F_0 can be a face of F_1, F_1 a fake of K, but F_0 cannot be a face of K. For a polytope, however, the converse of Lemma 4.4 is true (see Chapter II, Theorem 1.7).

Now, we will generalize Lemma 4.4.

4.5 Lemma. *If $F_1, \ldots, F_r$ are faces of a closed convex set K, then, $F := F_1 \cap \ldots \cap F_r$ is also a (possibly improper) face of K.*

Proof: Since being a face is not affected by doing so, we may assume $0 \in F$ (unless $F = \emptyset$ in which case there is nothing to prove).

Let $H_i = \{x \mid \langle x, u_i \rangle = 0\}$ be a supporting hyperplane of K such that $F_i = K \cap H_i, i = 1, \ldots, r$. By possibly changing signs of some of the u_i, we can arrange

$$K \subset H_i^- = \{x \mid \langle x, u_i \rangle \leq 0\}, \quad i = 1, \ldots, r.$$

We set $u := u_1 + \ldots + u_r$. If necessary, we can replace u_1 by $2u_1$ so that $u \neq 0$ can always be assumed. We find

$$\langle x, u \rangle = \langle x, u_1 \rangle + \ldots + \langle x, u_r \rangle \leq 0 \quad \text{for all } x \in K.$$

Therefore, $H := \{x \mid \langle x, u \rangle = 0\}$ is a supporting hyperplane of K. Moreover, $\langle x, u \rangle = 0$ is true if and only if $\langle x, u_1 \rangle = \ldots = \langle x, u_r \rangle = 0$. Hence,

$$x \in K \cap H \text{ if and only if } x \in (K \cap H_1) \cap \ldots \cap (K \cap H_r) = F.$$

4.6 Lemma.

1. *Suppose F is a face of the closed convex set K and $x, \tilde{x} \in$ rint F. Then, any supporting hyperplane of K at x also contains $\tilde{x}$.*

2. *If F, F' are faces of K and (rint F) $\cap$ (rint F') $\neq \emptyset$, then, $F = F'$.*

Proof:

1. (a) The line segment $[x, x']$ is properly contained in a line segment $[y, y'] \subset$ rint F. Should a supporting hyperplane at x not contain $\tilde{x}$ two of the points $\tilde{x}$ and $y, \tilde{y}$, would be separated, a contradiction.

2. (b) is obvious.

4.7 Definition. Let x be a point of the closed convex set K. We call

$$N(x) := -x + p_K^{-1}(x)$$

the *normal cone* of K at x.

4.8 Lemma. *$N(x)$ is a closed convex cone; it consists of 0 and all outer normals of K in x. If $x \in$ int K, then, $N(x) = \{0\}$.*

Proof: First, note that $N(x)$ is, indeed, a cone. From Lemmas 3.5 and 3.6, we deduce the second part of the lemma. $p_K^{-1}(x)$ and, hence, $-x + p_K^{-1}(x)$ is closed since p_K is continuous (Lemma 3.7). To show that $N(x)$ is convex, we arrange for $x = 0$ with a translation. Then, for $u, v \in N(0)$, we may assume $\langle K, u \rangle \leq 0$ and $\langle K, v \rangle \leq 0$, so that

$$\langle K, \lambda u + (1 - \lambda) v \rangle \leq 0 \qquad \text{for } 0 \leq \lambda \leq 1;$$

hence, $\lambda u + (1 - \lambda) v \in N(0)$.

□

4.9 Definition. Let σ be a cone. Then,

$$\check{\sigma} := \{y \mid \langle \sigma, y \rangle \geq 0\}$$

is called the *dual cone* of σ (Figure 7).

Lemma 4.8 implies Lemmas 4.10 and 4.11.

4.10 Lemma. *If σ is a cone with apex 0, then $N(0) = -\check{\sigma}(\check{\sigma}$ reflected in 0).*

4.11 Lemma. *Let F be a face of the closed convex set K. For $x, \tilde{x} \in$ rint F, $N(x) = N(\tilde{x})$.*

Proof: This follows readily from Lemma 4.6.

4.12 Definition. If F is a face of a closed convex set K and $x \in$ rint F, then $N(x)$ is denoted by $N(F)$ and is called the *cone of normals* of K in F.

4.13 Theorem. *Let K be a convex body in $\mathbb{R}^n$ and $x(F)$ one of the relative interior points of a face $F \neq \emptyset$ of K. Then, $\{$rint $N(x(F)) \mid F$ a face of $K\} = \{$ rint $N(F) \mid F$ a face of $K\}$ is a partition (disjoint covering) of $\mathbb{R}^n$.*

Proof: Let $0 \neq u \in \mathbb{R}^n$. Since K is bounded, there exists a hyperplane $H(\alpha, u) = \{z \mid \langle z, u \rangle = \alpha\}$ such that $K \subset H^-(\alpha, u)$. Put $H^- = \bigcap_\alpha H^-(\alpha, u)$, the intersection taken for all α, such that $K \subset H^-(\alpha, u)$. Clearly, H^- is again a closed half-space and $F := H \cap K \neq \emptyset$. For $x(F) \in$ rint F, $u \in$ rint $N(x(F))$; this is elementary in every plane passing through $x(F)$ and containing u; hence, it carries over the general situation. So, every $u \neq 0$ occurs in some cont rint $N(x(F))$. Also, the point 0 occurs in rint $N(x(K))$ since, for $x \in$ rint K, the cone $N(x)$ is a linear space ($= \{0\}$ if dim $K = n$).

Suppose $y \in$ rint $N(x(F_1)) \cap$ rint $N(x(F_2))$. Then, $p_K(y + x(F_1)) = x(F_1)$ and $p_K(y + x(F_2)) = x(F_2)$ so that, by Lemma 3.5, the supporting hyperplanes in $x(F_1)$ and $x(F_2)$ coincide. This implies $F_1 = F_2$.

□

4.14 Definition. $\Sigma(K)$ denotes the set of all cones $N(F)$ and is called the *fan* of K (see Figure 8).

Exercises

1. Let K be convex and closed, int $K \neq \emptyset$, and let L be an affine subspace such that $L \cap$ int $K = \emptyset, L \cap K \neq \emptyset$. Show that there exists a supporting hyperplane of K which contains L.

2. Characterize convex polytopes which have the same fan.

Support function and distance function

Now we will generalize the linear function $h_{\{a\}} := \langle a, \cdot \rangle$ for arbitrary compact subsets K of $\mathbb{R}^n$:

5.1 Definition. Let $K \subset \mathbb{R}^n$ be a nonempty convex body. The map

$$h_K : \mathbb{R}^n \to \mathbb{R} \text{ defined by } u \mapsto \sup_{x \in K} \langle x, u \rangle$$

is called the *support function* of K. The next statement is an obvious consequence of the definition.

5.2 Lemma. *If $K + a$ is a translate of the convex body K, then,*

$$h_{K+a}(u) = h_K(u) + \langle a, u \rangle \quad \text{for all } u \in \mathbb{R}^n.$$

Example 1. For $n = 1$, set $K = [c, d]$. Then

$$h_{[c,d]}(u) = \begin{cases} \langle d, u \rangle & \text{for } u \geq 0 \\ \langle c, u \rangle & \text{for } u \leq 0 \end{cases}$$

5.3. Lemma.

1. *For every fixed nonzero $u \in \mathbb{R}^n$, the hyperplane*

   $$H_K(u) := \{x \mid \langle x, u \rangle = h_K(u)\}, \qquad (*)$$

   is a supporting hyperplane of K.

2. *Every supporting hyperplane of K has a representation of the form $(*)$.*

Proof:

1. (1) Since K is compact and $\langle \cdot, u \rangle$ is continuous, for some $x_0 \in K$,

   $$\langle x_0, u \rangle = h_K(u) = \sup_{x \in K} \langle x, u \rangle.$$

   For an arbitrary $y \in K$, it follows that $\langle y, u \rangle \leq \langle x_0, u \rangle$; hence $K \subset H_K^-(u)$. This proves (1).

2. (2) Let $H = \{x \mid \langle x, u \rangle = \langle x_0, u \rangle\}$ be a supporting hyperplane of K at x_0. We choose $u \neq 0$ such that $K \subset H^-$. Then, $\langle x_0, u \rangle = \sup_{x \in K} \langle x, u \rangle = h_K(u)$ which implies (2).

□

5.4 Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex* if, for all $0 \leq \lambda \leq 1$ and $x, y \in \mathbb{R}^n$,

$$f(\lambda x + (1-\lambda)y \leq \lambda f(x) + (1-\lambda)f(y).$$

Note that if f is convex and L is an affine subspace of $\mathbb{R}^n$, then, $f|_L$ is also convex.

Example 2. For $n = 1$ and $x, y \in \mathbb{R}$, the graph $\Gamma(f)$ of a convex function f lies "below" the line-segment $[(x, f(x)), (y, f(y))]$ in $\mathbb{R}^2$. Hence for convex f, if $a \leq -1 < b < 0, f(b) = 1$, and $f(0) = 0$, then, $(a, f(a))$ and $(-b, -f(-b))$ are "above" the line through $(b, 1)$ and $(0, 0)$, so that $f(a) \geq -\frac{1}{b}$ and $f(b) \geq -f(-b)$.

5.5 Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *positive homogeneous* if, for any $\lambda \geq 0$ and $x \in \mathbb{R}^n$,

$$f(\lambda x) = \lambda f(x).$$

5.6 Lemma. *A positive homogeneous function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if*

$$f(x+y) \leq f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}^n \tag{1}$$

Proof: Let the positive homogeneous function f be convex. Then (1) follows from

$$\frac{1}{2}f(x+y) = f(\frac{1}{2}x + \frac{1}{2}y) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

Conversely, if (1) holds for f, then, for $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)y) \leq f(\lambda x) + f((1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y),$$

so f is convex.

□

5.7 Lemma.

1. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if, for every convex combination $x = \lambda_0 x_0 + \ldots + \lambda_n x_n, \lambda_0 \geq 0, \ldots, \lambda_n \geq 0, \lambda_0 + \ldots + \lambda_n = 1$ of points $x_0, \ldots, x_n$*

$$f(x) \leq \lambda_0 f(x_0) + \ldots + \lambda_n f(x_n). \tag{1}$$

2. *Every convex function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous.*

3. *$f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if $\Gamma^+(f) := \{(x, \xi) \mid x \in \mathbb{R}^n, \xi \in \mathbb{R}, f(x) \leq \xi\}$ is a closed and convex subset of $\mathbb{R}^{n+1}$.*

4. *A positive homogeneous function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if $\Gamma^+(f)$ is a closed convex cone.*

Proof:

1. (1) If eq. (1) is true, we obtain, for $x_1 = \ldots = x_n$ (using $1 - \lambda_0 = \lambda_1 + \ldots + \lambda_n$),

$$f(\lambda_0 x_0 + (1 - \lambda_0)x_1) \leq \lambda_0 f(x_0) + (1 - \lambda_0) f(x_1),$$

   so that f is convex.

   If conversely, f is convex, we proceed by induction and assume f satisfies (1) (with n replaced by $n-1$) on each $(n-1)-$dimensional affine subspace of $\mathbb{R}^n$. Then, for $\lambda_0 < 1$ and $y := (1 - \lambda_)^{-1}(\lambda_1 x_1 + \ldots + \lambda_n x_n) = (\lambda_1 + \ldots + \lambda_n)^{-1}(\lambda_1 x_1 + \ldots + \lambda_n x_n)$, we find

$$\begin{aligned}
&f(\lambda_0 x_0 + \ldots + \lambda_n x_n) = f(\lambda_0 x_0 + (1 - \lambda_0)y) \\
&\leq \lambda_0 f(x_0) + (1 - \lambda_0) f(y) \\
&\leq \lambda_0 f(x_0) + (\lambda_1 + \ldots + \lambda_n)\left(\sum_{i=1}^{n} (\lambda_1 + \ldots + \lambda_n)^{-1} \lambda_i f(x_i)\right) \\
&= \lambda_0 f(x_0) + \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)\,,
\end{aligned}$$

   so that (1) follows.

2. (2) Given a point x_0 in $\mathbb{R}^n$, we consider a regular $n-$simplex $T :=$conv $\{x_1, \ldots, x_{n+1}\}$ which possesses x_0 as center of gravity and for which $\|x_1 - x_0\| = \ldots = \|x_{n+1} - x_0\| = 1$. We set $d := \max\{|f(x_1) - f(x_0)|, \ldots, |f(x_{n1}) - f(x_0)|\}$. Let x lie in a δ_0-neighborhood $U_{\delta_0}(x_0)$ of x_0 such that $U_{\delta_0}(x_0) \subset T$. Since T is covered by $n-$simplices $T_i :=$conv $\{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}\}, i = 1, \ldots, n+1$, we may assume x to line in one of the T_i, say in T_{n+1}, $x = \lambda_0 x_0 + \ldots + \lambda_n x_n\,, \lambda_0 \geq 0, \ldots, \lambda_n \geq 0, \lambda_0 + \ldots + \lambda_n = 1$. Clearly, $\lambda_i < \delta_0 \leq 1\,, i = 1, \ldots, n$. We may assume $f(x) \geq 0$ in T (up to adding a constant). Given $\epsilon > 0$, we may choose $\delta := \frac{\epsilon}{n(d+1)}$ and obtain (using (1) and assuming $\delta \leq \delta_0$)

$$\begin{aligned}
|f(x) - f(x_0)| &\leq |\lambda_0 f(x_0) + \ldots + \lambda_n f(n) - f(x_0)| \\
&= |\lambda_1(f(x_1) - f(x_0)) + \ldots + \lambda_n(f(x_n) - f(x_0))| \\
&\leq (\lambda_1 + \ldots + \lambda_n)d < n\delta(d+1) = \epsilon.
\end{aligned}$$

   Therefore, f is continuous.

3. (3) let f be convex. Given $(x, \xi), (y, \eta) \in \Gamma^+(f), 0 \leq \alpha \leq 1\,$,

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha) f(y) \leq \alpha \xi + (1 - \alpha)\eta;$$

   hence,

$$\alpha(x, \xi) + (1 - \alpha)(y, \eta) = (\alpha x + (1 - \alpha)y, \alpha\xi + (1 - \alpha)\eta) \in \Gamma^+(f).$$

   Therefore, $\Gamma^+(f)$ is convex. From (2), it readily follows that $\Gamma^+(f)$ is also closed. The arguments may, obviously, be reversed.

4. (4) Consider if f is homogeneous and convex, the closed set $\Gamma^+(f)$ is a cone. If, conversely, $\Gamma^+(f)$ is a closed and convex cone, f is homogeneous, and by (3), convex.

□

Remarks.

1. By Caratheodory 's theorem, in (a) we may choose x to be a convex combination of an arbitrary number of points.

2. If, in the definition of a convex function, $\mathbb{R}^n$ is replaced by a closed convex subset of $\mathbb{R}^n$, (2) and (3) need no longer be true. Example: Let the subset be the closed unit ball B of $\mathbb{R}^n$, and let $f(x) = 0$ for $x \in \text{int } B, f(x) = 1$ for $x \in \partial B$.

5.8 Lemma. *The support function h_K of a convex body K is positive homogeneous and convex.*

Proof: Let $\lambda \geq 0$. It's easy to see

$$h_K(\lambda u) = \sup_{x \in K} \langle x, \lambda u \rangle = \lambda \sup_{x \in K} \langle x, u \rangle = \lambda h_K(u).$$

Hence, h_K is positive homogeneous.

From $\langle x, u \rangle \leq h_K(u), \langle x, v \rangle \leq h_K(v) \ \forall x \in K$, we obtain

$$\langle x, u + v \rangle \leq h_K(u) + h_K(v) \qquad \text{for all } x \in K.$$

Hence,

$$h_K(u + v) = \sup_{x \in K} \langle x, u + v \rangle \leq h_K(u) + h_K(v).$$

Therefore, by Lemma 5.6, h_K is convex.

□

5.9 Lemma. *h_K is linear on each cone of the fan $\Sigma(K)$ of K.*

Proof: All points u in a fixed cone σ of $\Sigma(K)$ have the same nearest point $x_0 := p_K(u)$. As in the proof of Lemma 5.3(b) we obtain

$$h_K|_\sigma = \langle x_0, \cdot \rangle|_\sigma.$$

5.10 Definition. Let K be an $n-$dimensional convex body in $\mathbb{R}^n$, and let $0 \in \text{int } K$. Then we can denote the map as

$$d_K : \mathbb{R}^n \to \mathbb{R}$$

defined by

$$d_K(\lambda\tilde{x}) := \lambda\,, \qquad \text{for } \bar{x} \in \partial K \text{ and } \lambda \geq 0\,,$$

is called the *distance function* of K.

We can show that d_K is well defined (part (b) of the following lemma).

5.11 Lemma. *Let K be an $n-$dimensional convex body in $\mathbb{R}^n$.*

1. If a line g intersects ∂K in three different points, then, g is contained in a supporting hyperplane of K, so, in particular, $g \cap \text{int } K = \emptyset$.
2. Any ray emanating from a point in int K intersects a K in one and only one point.

Proof:

1. Let $A, B, C \in g \cap \partial K$, and let B lie between A and C. We consider a supporting hyperplane $H = \{x \mid \langle x, u\rangle = c\}$ of K in B. If H did not contain both A and C, it would separate these points properly, which contradicts the definition of a supporting hyperplane.
2. Let $y \in \text{int } K, \sigma$ be a ray emanating from y, and h be the line that contains σ. The intersection $h \cap K$ is a convex body, hence, a line segment $[y_0, y_1]$. Either y_0 or y_1 equals $\sigma \cap \partial K$.

□

5.12 Lemma. *The distance function d_K is positive homogeneous and convex.*

Proof: By definition, d_K is positive homogeneous.

To prove convexity, let $d_K(x) = \lambda, d_K(y) = \mu$. If $\lambda = 0$ or $\mu = 0$, then $x = 0$ or $y = 0$, and there is nothing to prove. So, let us consider $\lambda, \mu \neq 0$. Take $\delta := \frac{\mu}{\lambda+\mu}$, we obtain $(1-\delta)\bar{x} + \delta\bar{y} \in K$, for $\lambda\bar{x} = x, \mu\bar{y} = y$, hence,

$$1 \geq d_K((1-\delta)\bar{x} + \delta\bar{y}) = d_K(\frac{\lambda}{\lambda+\mu}\bar{x} + \frac{\mu}{\lambda+\mu}\bar{y}) = d_K(\frac{1}{\lambda+\mu}(x+y))$$

$$= \frac{1}{\lambda+\mu} d_K(x+y)\,,$$

hence, $d_K(x+y) \leq \lambda + \mu = d_K(x) + d_K(y)$. So d_K is convex by Lemma 5.6.

□

5.13 Definition. A convex body K is called *centrally symmetric* if it is mapped onto itself by a reflection in a point c (which assigns to each $x = c+(x-c)$ the point $c-(x-c) = 2c-x$. We call c the *center* of K.

From the above lemmas, we can derive Theorem 5.14.

5.14 Theorem. *Let K be a centrally symmetric convex body with $0 \in int\ K$ as its enter. Then, d_K defines a norm on the vector space $\mathbb{R}^n$, that is, a map*

$$d_K = \|\cdot\| : \mathbb{R}^n \to \mathbb{R}$$

satisfying, for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

- $\|x\| = 0$ *if and only if* $x = 0$,
- $\|\lambda x\| = |\lambda| \cdot \|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$.

Example 3. The "maximum norm" in $\mathbb{R}^2$ is of the form

$$d_K(x) := \max\{|x_1|, |x_2|\}$$

where $x = (x_1, x_2)$ and K is the square with vertices $(1,1), (1,-1), (-1,1), (-1,-1)$.

Example 4. Consider the "*Manhattan norm*" $d_{K'}(x) := |x_1| + |x_2|$ where K' is the square with vertices $(1,0), (0,1), (-1,0), (0,-1)$.

In the following section, I will provide explanation into how these norms in the above examples are interconnected.

Exercises

Polar bodies

Let us consider the polarity π in $\mathbb{R}^n$ with respect to the unit sphere $S := \{x \mid \langle x, x \rangle = 1\}$. It assigns to every affine subspace W of $\mathbb{R}^n$ with $0 \notin W$ a subspace $\pi(W)$ of of $\mathbb{R}^n$ of dimension $n-1-$ dim W : If $0 \neq u$ is a point in $\mathbb{R}^n$, then,

$$\pi(u) = H_u := \{x \mid \langle x, u \rangle = 1\}.$$

If the affine subspaces U and V which generate W are not parallel and if W does not contain 0, then, $\pi(W) = \pi(U) \cap \pi(V)$. Note that $\pi \circ \pi$ is the identity.

The exceptional role of the point 0 can be avoided by going over to the projective extension of $\mathbb{R}^N$ by adding a "hyperplane at infinity", H_∞. Then, $\pi(0) = H_\infty$. That will be needed, for example, in Lemma 3.

6.1 Definition. Let $0 \in$ int K, where K is a convex body. Then, for $u \neq 0$, the half-spaces H_u^- which contain 0 and, for $H_0^- := \mathbb{R}^n$,

$$K^* := \bigcap_{u\in K} H_u^-$$

is called the *polar body* of K. Clearly then, we see that $0 \in$int K^* and $K^* = \bigcap_{u\in\partial K} H_u^-$, since $0 \in$ int K.

6.2 Definition. We will represent the points of $\mathbb{R}^n \cup H_\infty$ by the one-dimensional subspaces of $\mathbb{R}^{n+1}$ such that the points of H_∞ are spanned by vectors $(0, \ldots, 0, \xi)$, $\xi \neq 0$. Then, a linear transformation of $\mathbb{R}^{n+1}$ up to multiplication by a nonzero factor is called a *projective transformation of* $\mathbb{R}^n \cup H_\infty$. It is called *permissible* with respect to the convex body $K \subset \mathbb{R}^n \cup H_\infty$, if H_∞ is mapped onto a hyperplane disjoint from K.

6.3 Lemma. *If the convex body K is so translated to $\tau(K)$ that 0 remains in the interior, then, $(\tau(K))^*$ is obtained from K^* by a permissible projective transformation.*

Proof: This follows from general facts on projective transformations.

□

6.4 Theorem. *Let K be a convex body with $0 \in$ int K. Then,*

1. $K^{**} = K$;
2. The distance function of K equals the support function of K^*, and, conversely

$$d_K = h_{K^*} \qquad d_K^* = h_K.$$

Proof.

- By definition of H_u, for every $u \neq 0$ of K,

$$H_u^- = \{x \mid \langle u, x\rangle \leq 1\}$$

Therefore, using the obvious notation of $\langle K, x\rangle \leq 1$, we can write K^* as

$$K^* = \{x \mid \langle K, x\rangle \leq 1\} \quad \text{and} \quad K^{**} = \{y \mid \langle K^*, y\rangle \leq 1\}.$$

If $y \in K$, then, the definition of K^* yields $\langle y, K^*\rangle \leq 1$ and, thus, $K \subset K^{**}$. Suppose $K \neq K^{**}$. Then, let $x \in K^{**}\backslash K$. For

$$x' := p_K(x) \quad \text{and} \quad u := \frac{x - x'}{\langle x', x - x'\rangle} ,$$

Invoking Lemma 3.5 yields

$$x \in H_u^+ \backslash H_u\,, \qquad \text{but also } K \subset H_u^-\,,$$

whence $u \in K^*$. Since $x \in K^{**}$, it follows that $\langle u, x\rangle \leq 1$,i.e., $x \in H_u^-$, a contradiction. Thus we've proved part (a), but need two supporting lemmas to prove (b) first.

□

6.5 Lemma. *Let K_1, K_2 be convex bodies such that $0 \in$ int K_1 and $K_1 \subset K_2$. Then, $K_2^* \subset K_1^*$.*

Proof: If $y \in K_2^*$, then, $\langle K_2, y\rangle \leq 1$, hence, in particular, $\langle K_1, y\rangle \leq 1$. This implies $y \in K_1^*$.

6.6 Lemma. *If $x \in \partial K$, $0 \in$ int K, then, H_x is a supporting hyperplane of K^*.*

Proof: We know that $K^* = \bigcap_{x\in\partial K} H_x^-$. For every $x \in \partial K$, there exists a $\beta_x \in \mathbb{R}_{\geq 1}$ such that $H_{\beta,x}$ is a supporting hyperplane of K^*. Thus, $\tilde{K} := \operatorname{conv}(\{\beta_x x \mid x \in \partial K\})$ includes K, obtaining

$$\tilde{K}^* = \bigcap_{y\in\partial\tilde{K}} H_y^- = \bigcap_{x\in\partial K} H_{\beta,x}^- \supset K^* = \bigcap_{x\in\partial K} H_x^-.$$

Since, obviously, $H_{\beta_x,x}^- \subset H_x^-$, we find that $\beta_x = 1$ for every $x \in \partial K$.

□

Proof of (b) in Theorem 6.4. Let $u \in \mathbb{R}^n\backslash\{0\}$. We may assume $u \in \partial K$, hence, $d_K(u) = 1$. By Lemma 6.6, H_u is a supporting hyperplane of K^*, and we obtain $h_{K^*}(u) = 1$ from Lemma 5.3.

□

6.7 Theorem. *Let K be a convex body in $\mathbb{R}^n$ with $0 \in$ int K. Set $K_+ := \Gamma^+(d_K) \subset \mathbb{R}^{n+1}$ (see Lemma 5.7) and $H := \{(x,1) \mid x \in \mathbb{R}^n\}$. Then,*

1. ∂K_+ is the graph of d_K in $\mathbb{R}^{n+1}$.
2. $K_+ \cap H$ is a translate of K.
3. $K_+^* \cap H$ is a translate of K^*.
4. K_+, K_+^* are cones with apex 0 in $\mathbb{R}^{n+1}$.

6.8 Theorem. *Every positive homogeneous and convex function $h : \mathbb{R}^n \to \mathbb{R}$ is the support function $h = h_K$ of a unique body K (whose dimension is possibly less than n).*

Proof. Let us write $\mathbb{R}^n = U \bigoplus U^\perp$, where U is the maximal linear subspace of $\mathbb{R}^n$ on which h is linear. Then, there exists $a \in U$ such that, for $(x, x') \in U \bigoplus U^\perp$,

$$h(x, x') = \langle x, a\rangle + h|_{U^\perp}(x'). \qquad (*)$$

Moreover, $\Gamma^+(h|_{U^\perp})$ is a cone with apex 0 in $U^\perp \bigoplus \mathbb{R}$ (see Lemma 5.7). Thus, there exists some $b \in U^\perp$ such that the hyperplane $H := \{(y, \langle y, b\rangle)|y \in U^\perp\}$ in $U^\perp \bigoplus \mathbb{R}$ intersects $\Gamma^+(h|_{U^\perp})$ only in the apex. Now the set

$$K_0 + (0, 1) := (U^\perp \times \{1\}) \cap \Gamma^+(h|_{U^\perp} - \langle \cdot, b\rangle)$$

is a convex body and, by Lemma 5.2, $h|_{U^\perp} - \langle \cdot, b\rangle$, the support function of $K_0 - b$. Finally, $(*)$ and Lemma 5.2 yield that h is the support function of $K := K_0 - b + a$.

□

Exercise

1. Let K be an unbounded closed convex set, dim $K = n$, and let $0 \in$ int K. We set $K^* := \bigcap_{u \in K} H_u^-$ where $H_0^- := \mathbb{R}^n$.
   - Show that K^* is a convex body,
   - Must $K^{**} = K$?

2 Combinatorial theory of polytopes and polyhedral sets

2.1 The boundary complex of a polyhedral set

We will now turn to the specific properties of convex polytopes, or, briefly, polytopes. In 1.1 we introduced these as convex hulls of finite point sets in $\mathbb{R}^n$. Our first aim is to show that, equivalently, convex polytopes can be defined as bounded intersections of finitely many half-spaces.

1.1 Theorem. *Each polytope possesses only finitely many faces; they, too, are polytopes.*

Proof: Let $P = \text{conv}\{x_1, \ldots, x_r\}$, and let $F := P \cap H$ be a face where $H = \{x \mid \langle x, a\rangle = \alpha\}$ is a supporting hyperplane of P such that $P \subset H^-$. We may assume the following

$$x_1, \ldots, x_s \in H; \quad x_{s+1}, \ldots, x_r \in \text{ int } H^-$$

and find

$$\langle x_i, a\rangle = \alpha \qquad \text{for } i = 1, \ldots, s$$
$$\langle x_i, a\rangle = \alpha - \beta_i, \beta_i > 0 \qquad \text{for } i = s+1, \ldots, r.$$

Then, for

$$x = \lambda_1 x_1 + \ldots + \lambda_r x_r, \quad \lambda_1 + \ldots + \lambda_r = 1, \;\; \lambda_j \geq 0, \;\; j = 1, \ldots, r\,,$$
$$\langle x, a\rangle = \sum_{i=1}^{r} \lambda_i \langle x_i, a\rangle = \sum_{i=1}^{r} \lambda_i \alpha - \sum_{i=s+1}^{r} \lambda_i \beta_i = \alpha - \sum_{i=s+1}^{r} \lambda_i \beta_i.$$

Therefore, $x \in H$ if and only if $\sum_{i=s+1}^{r} \lambda_i \beta_i = 0$, which, in turn, is equivalent to $\lambda_{s+1} = \ldots = \lambda_r = 0$. So, x is a convex combination of $x_1, \ldots, x_s$. Hence $H \cap P = \text{conv}\{x_1, \ldots, x_s\}$ is a polytope.

Since only finitely many convex hulls of elements of $\{x_1, \ldots, x_r\}$ exist, the theorem follows.

□

1.2 Krein-Milman Theorem. *Each polytope P is the convex hull of its vertices, that is,*

$$P = \text{ conv(vert } P).$$

Proof: Obviously, we can see that conv(vert P) $\subset P$. For the opposite inclusion, we may assume that $P = \text{conv}\{x_1, \ldots, x_r\}$ and $x_i \notin \text{conv}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r\} =: P_i$ for $1 \leq i \leq r$. Denote by $q_i := p_{P_i}(x_i)$ to be the image of x_i under the nearest point map p_{P_i} with respect to P_i. By I, Lemma 5.3, the hyperplane H_i through q_i with normal $x_i - q_i$ is a supporting hyperplane of P_i. We translate H_i by adding $x_i - q_i$ and so obtain a supporting hyperplane H_i' of P for which

$$\{x_i\} = H_i' \cap P$$

Therefore x_i is a vertex of P. This implies $P \subset$ conv(vert P). Hence, the theorem obviously follows.

□

1.3 Definition. The intersection of finitely many closed half-spaces in $\mathbb{R}^n$ is called a *polyhedral* set.

1.4 Theorem. *Every polytope P is a bounded polyhedral set.*

Proof: We may assume that aff $P = \mathbb{R}^n$. Now let $F_i := P \cap H_i$ to be the facets of P(($n-1$)$-$dimensional faces), and let $P \subset H_i^-, i = 1, \ldots, s$.

Obviously, P is contained in

$$\bigcap_{i=1}^{s} H_i^- =: P'.$$

Suppose then that $x_0 \in P' \backslash P$. Consider the union $\mathcal{A}$ of all affine subspaces of $\mathbb{R}^n$ spanned by x_0 and at most $n-1$ vertices of P. Since $\mathcal{A}$ has no interior points, there exists

$$x \in (\text{int P}) \backslash \mathcal{A}.$$

The line segment $[x, x_0]$ is not contained in $\mathcal{A}$ and intersects ∂P in a point y. Since ∂P is the union of all (proper) faces of P (I, Theorem 3.9), y is contained in a face F. From dim $F < n-1$ would follow $x \in \mathcal{A}$, a contradiction. Therefore F is a facet, say F_0' and $y \in$ rint F. But, then, aff F_0 would be one of the hyperplanes H_i, $i \in \{1, \ldots, s\}$, and so, $x_0 \notin P'$, a contradiction to the initial assumption.

□

1.5 Theorem. *Every bounded polyhedral set is a polytope.*

Proof: We will proceed by induction on dim P, $P := H_1^- \cap \ldots \cap H_i^-$. Let us assume that each of the (proper) faces $F_j := H_j \cap P$ is a polytope. Replacing $\mathbb{R}^n$ by aff P we may assume that P is of maximal dimension. Obviously,

$$\text{conv}\left(\bigcup_{j=1}^{s} F_j\right) \subset P;$$

it suffices, thus, to show the opposite inclusion for int P. For $x \in$ int P, fix a ray σ emanating from x not parallel to any H_j for $j = 1, \ldots, s$. Then, by I, Lemma 5.11, $\sigma \cap \partial P$ consists of one point x_σ. Since $\partial P \subset \cup_{j=1}^{s} F_j$, the point x_σ is contained in a face, say F_{j_σ}. The analogous statement holds for the ray opposite to σ. Since $x \in [x_\sigma, x_\tau]$, we find $x \in$ conv$(F_{j_\sigma} \cup F_{j_{\tau]}})$, and, then,

$$\text{int } P \subset \text{ conv } \left(\bigcup_{j=1}^{s} F_j\right).$$

□

We may then summarize Theorems 1.4 and 1.5 as follows:

$$\text{polytopes} = \text{bounded polyhedral sets}$$

1.6 Corollary. *Any affine subspace L of $\mathbb{R}^n$ intersects a given polyhedral set (polytope) P in a polyhedral set (polytope).*

We are now ready to prove the converse of I, Lemma 4.4, in the case of polytopes.

1.7 Theorem. *Let P be a polyhedral set. If F_1 is a face of P and F_0 is a face of F_1, then, F_0 is a face of P.*

Proof: First, let us assume P to be bounded, that is, a polytope P and vertices $P =: \{x_1, \dots, x_m\}$. We may assume that $x_1 = 0 \in F_0 \neq F_1$. There are linearly independent u_0, u_1 such that, for $H_i := \{x \mid \langle x, u_i \rangle = 0\}, i = 0, 1,$

$$F_0 = H_0 \cap F_1, \quad F_1 \subset H_0^-$$
$$F_1 = H_1 \cap P, \quad P \subset H_1^-.$$

We denote by $x_2, \dots, x_s$ the vertices of $P \backslash F_1$, by $x_{s+1}, \dots, x_t$ those of $F_1 \backslash F_0$. For $i = 2, \dots, s$, there exist points u_i such that

$$H_i := lin(\{x_i\} \cup (H_0 \cap H_1)) = \{x \mid \langle x, u_i \rangle = 0\}.$$

All u_i lie in the plane $(H_0 \cap H_1)^\perp$; hence, we may assume that $F_1 \subset \bigcap_{i=2}^s H_i^-$ and that all u_i, considered as points, lie on the line g through u_0 and u_1,

$$u_i = u_0 + \alpha_i(u_1 - u_0), \qquad i = 2, \dots, s.$$

The u_i's even lie on the ray of g emanating from u_1 and including u_0, since $\alpha_i \in \mathbb{R}_{<1}$. From $x_j \in H_i^-$, for $j \in \{s+1, \dots, t\}$, we see that

$$0 > \langle x_j, u_i \rangle = (1 - \alpha_i)\langle x_j, u_0 \rangle.$$

Since $F_1 \subset H_0^-$ implies $\langle x_j, u_0 \rangle < 0, (1 - \alpha_i) > 0$. Hence, there exists a point $u \in g$ separating u_1 from $\{u_2, \dots, u_s\}$ properly, that is,

$$u = \lambda_i u_1 + (1 - \lambda_i)u_i, \quad \text{for some } 0 < \lambda_i < 1, \ i = 2, \dots, s.$$

The hyperplane $H := \{x \mid \langle x, u \rangle = 0\}$ is a supporting hyperplane of P with $H \cap P = F_0$. For $x_j \in F_1$, we obtain

$$\langle x_j, u \rangle = \lambda_i \langle x_j, u_1 \rangle + (1 - \lambda_i)\langle x_j, u_i \rangle = (1 - \lambda_i)\langle x_j, u_i \rangle \leq 0\,,$$

since $F_1 \subset H_i^-$. Thus, $\langle x_j, u \rangle = 0$ if and only if $x_j \in F_0 \subset H_0 \cap H_1$. For $x_i \in$ vertices $P \backslash F_1$, $\langle x_i, u_1 \rangle < 0$ and, thus

$$\langle x_i, u \rangle = \lambda_i \langle x_i, u - 1 \rangle + (1 - \lambda_i)\langle x_i, u_i \rangle = \lambda_i \langle x_i, u_1 \rangle < 0\,,$$

which implies $P \subset H^-$ and $P \cap H = F_0$.

If P is not a polytope, we choose a sufficiently large $n-$simplex S so that int S intersects each face of P. Then, all bounded faces of P are contained in int S. If F is an unbounded face of P, we find that $F = P \cap H$, H is a supporting hyperplane of P, if and only if $F \cap S = P \cap S \cap H$. Each face F of P intersects $P \cap S$ in a face $F' := P \cap S \cap F$ of $P \cap S$ such that dim $F =$ dim F'. So, the theorem readily follows from its validity for $P \cap S$.

□